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Analytical Analysis of Fractional Order Swift-Hohenberg Equations

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***Abstract*—** This study utilizes the New Approximate Analytical Method (NAAM) to solve fractional order Swift-Hohenberg (S-H) equations, which model complex systems with memory and hereditary effects. By deriving approximate analytical solutions for both linear and nonlinear cases, NAAM simplifies the analytical analysis of these equations, particularly those involving fractional derivatives. A comparative evaluation with other traditional analytical and numerical methods highlights the NAAM's effectiveness and accuracy, advancing the understanding of fractional systems and providing improved tools for addressing complex mathematical models.

***Keywords*—** New Approximate Analytical Method (NAAM), Swift-Hohenberg (S-H) Equations, Fractional Order Differential Equations, Analytical Solution, Numerical and Graphical Analysis.

I. INTRODUCTION

The Partial Differential Equations (PDEs) extend ODEs to multiple variables and have been pivotal in modeling physical and biological systems. PDEs emerged in the 18th century, with contributions from d'Alembert, who formulated the wave equation, and Fourier, who developed the heat equation and introduced Fourier series for solving PDEs. The 19th century saw further advances, such as Poisson's equation in electrostatics and fluid dynamics, and Cauchy's theorem on the existence and uniqueness of solutions to PDEs. The 20th century witnessed significant developments with Maxwell's equations in electromagnetism and the introduction of Sobolev spaces for handling generalized solutions. Analytical techniques like separation of variables and Green's functions, along with numerical methods such as finite difference, finite element, and spectral methods, are commonly used to solve PDEs [1-3]. Applications span physics, engineering, biology, and finance, modeling processes like wave propagation, heat conduction, and the behavior of financial markets.

Fractional Differential Equations (FDEs) generalize classical calculus to non-integer orders, modeling systems with memory effects and anomalous behaviors. Fractional calculus originated in the 17th century, with key contributions from Leibniz, Fourier, and Liouville. In the 20th century, mathematicians like Lévy and Kolmogorov explored its connections to stochastic processes and fractals. Solving FDEs often requires methods like the Laplace transform, the Mittag-Leffler function, or numerical approaches such as finite difference methods [4]. FDEs are applied in various fields, including physics, engineering, biology, and finance, providing models for systems with memory and complex behaviors.

The heat equation, formulated by Fourier, describes heat distribution over time and has broad applications across physics and engineering. Solving the heat equation involves analytical methods such as Fourier series and numerical techniques like finite difference and finite element methods. The fractional heat equation extends the classical model to account for anomalous diffusion processes [5]. Solving fractional heat equations is more challenging and often requires numerical methods.

The Swift-Hohenberg equation models pattern formation, such as rolls and hexagons, in systems like Rayleigh-Bénard convection. Its extension to fractional orders allows for modeling systems with memory effects and anomalous diffusion. Solving the fractional-order Swift-Hohenberg equation poses challenges due to the complexity of fractional derivatives, with numerical methods being commonly used [6]. This equation has applications in fields such as material science, biology, and geophysics, where memory effects play a critical role in system behavior.

II. NEW APPROXIMATE ANALYTICAL METHOD (NAAM)

Here we are going to recap a general methodology of NAAM [7-8].

Consider the general nonlinear time fractional order Swift-Hohenberg model as:

$$P_\tau^\alpha(\psi,\tau) = LP(\psi,\tau) + NP(\psi,\tau) + g(\psi,\tau), \quad \alpha \in (0,1]. \quad (1)$$

Here by the initial value

$$\mathrm{P}(\psi,0) = \mathrm{P}(\psi),$$

where symbols "L" and "N" are linear and nonlinear operators respectively.

For further procedure of computing, we are defining some other basic results.

Lemma [9]

For $\mathrm{P}(\psi,\tau) = \sum_0^\infty \mathrm{v}^{\mathrm{k}}\,\mathrm{P}_{\mathrm{k}}\,(\psi,\tau)$, the operator $\mathrm{LP}(\psi,\tau)$ is linear and having the property as:

$$LP(\psi,\tau) = L\left(\sum_0^\infty v^k\, P_k\,(\psi,\tau)\right) = \sum_0^\infty L(v^k\ P_k\,(\psi,\tau)). \quad (2)$$

Lemma [10]

Let $\mathrm{P}(\psi,\tau) = \sum_0^\infty \mathrm{P}_{\mathrm{k}}\,(\psi,\tau)$. By using the parameter λ, we will define $P_\lambda(\psi,\tau) = \sum_0^\infty \lambda^{\mathrm{k}}\,\mathrm{P}_\lambda\,(\psi,\tau)$. Then here the operator $NP(\psi,\tau)$ is nonlinear and satisfy the below mentioned property:

$$N(P_\lambda) = N\left(\sum_0^\infty \lambda^k\ P_k\,(\psi,\tau)\right) = \sum_0^\infty\left[\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_0^\infty \lambda^k\ P_k\,(\psi,\tau)\right)\right]_{\lambda=0}\right]\lambda^n. \quad (3)$$

Definition [10]

The polynomial $Q_k = Q_k(u_0,u_1,u_2,\dots,u_k)$ is defined as

$$Q_k = Q_k(u_0,u_1,u_2,\dots,u_k) = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N\left(\sum_0^\infty \lambda^k\, Q_k\,(\psi,\tau)\right)\right]_{\lambda=0}. \quad (4)$$

Definition [9]

If $\mathrm{Q}_{\mathrm{n}} = \mathrm{Q}_{\mathrm{n}}(\mathrm{u}_0,\mathrm{u}_1,\mathrm{u}_2,\cdots,\mathrm{u}_{\mathrm{n}})$, then by the above results of equation (4) of nonlinear operator $N(Q_\lambda)$ is expressed as

$$N(Q_\lambda) = \sum_0^\infty \lambda^k\, Q_k. \quad (5)$$

Lemma [10]

Let $\mathrm{g}(\psi,\tau), \mathrm{P}(\psi,\tau)$ be defined for $\mathrm{q}-1 < \alpha \leq \mathrm{q}$ in equation (1). Then the model (1) provides the unique solution in the form

$$P(\psi,\tau) = g_\tau^{-\alpha}(\psi,\tau) + P(\psi) + \sum_{k=1}^\infty\left[L_\tau^{-\alpha}(P_{k-1}) + Q_{(k-1)\tau}^{-\tau}\right], \quad (6)$$

where, $\mathrm{L}_\tau^{-\alpha}(\mathrm{P}_{\mathrm{k}-1})$ and $\mathrm{Q}_{(\mathrm{k}-1)\tau}^{-\alpha}$ represent the fractional partial integrals of order α for $\mathrm{L}(\mathrm{P}_{\mathrm{k}-1})$ and $\mathrm{Q}_{\mathrm{k}-1}$ with respect to τ.

Proof

Consider the solution of physical model $p(\psi,\tau)$ obtained by using the following expansion

$$P(\psi,\tau) = \sum_{k=0}^\infty P_k(\psi,\tau). \quad (7)$$

To solve the model with initial source (1), we consider

$$P_\tau\lambda^\alpha\,(\psi,\tau) = \lambda[LP(\psi,\tau) + NP(\psi,\tau) + g(\psi,\tau)], \quad \lambda \in [0,1], \quad (8)$$

along with initial condition

$$\mathrm{P}(\psi,0) = \mathrm{P}(\psi).$$

Furthermore, we approximate the solution of equation (6) as

$$P_\lambda(\psi,\tau) = \sum_0^\infty P_\lambda(\psi,\tau)\,. \quad (9)$$

Using the Riemann-Liouville partial fractional-integral operator of order α with respect to τ on both sides of the equation (8) and thus applying here the arguments of the R-Liouville operator of fractional order, we obtain

$$P_\lambda(\psi,\tau) = P(\psi,0) + \lambda I_\tau^\alpha\,[LP(\psi,\tau) + NP(\psi,\tau) + g(\psi,\tau)]. \quad (10)$$

By inserting the initial condition, equation (10) will be

$$P_\lambda\,(\psi,\tau) = P(\psi) + \lambda\, I_\tau^\alpha\,[LP(\psi,\tau) + NP(\psi,\tau) + g(\psi,\tau)]. \quad (11)$$

Substituting equation (9) into equation (11), we get

$$\sum_{k=0}^\infty \lambda^k\, P(\psi,\tau) = P(\psi) + \lambda[g(\psi,\tau)] + \lambda\, I_\tau^\alpha\,\left[L\left(\sum_{k=0}^\infty \lambda^k\, P(\psi,\tau)\right) + N\left(\sum_{k=0}^\infty \lambda^k\, P(\psi,\tau)\right)\right]. \quad (12)$$

Using equation (2) and equation (5), equation (12) becomes

$$\sum_{k=0}^\infty \lambda^k\, P\_\lambda(\psi,\tau) = P(\psi) + \lambda[g(\psi,\tau)] + \lambda\, I_\tau^\alpha\left[L\left(\sum_{k=0}^\infty \lambda^k\lambda^k P_k\right)\right] + \lambda\, I_\tau^\alpha\left[N\left(\sum_k^\infty \lambda^k Q_n\right)\right]. \quad (13)$$

By coefficient comparing at unique powers of λ in result equation (13), we get here the series form

$$\mathrm{P}_0(\psi,\tau) = \mathrm{P}(\psi,0) = \mathrm{P}(\psi),$$

$$\mathrm{P}_1(\psi,\tau) = \mathrm{g}(\psi,\tau) + \mathrm{L}_\tau^{-\alpha}\,\mathrm{P}_0 + \mathrm{Q}_{\tau_{(\mathrm{k}-1)}}^{-\alpha},$$

$$\vdots$$

$$P_r(\psi,\tau) = L_\tau^{-\alpha}\,P_{r-1} + Q_{\tau_{(r-1)}}^{-\alpha}, \quad \alpha = 2,3,\cdots \quad (14)$$

The final recursive scheme provide series form solution with fractional order. The obtained result then check for validity with exact solution for integer order.

III. SOLUTION OF LINEAR FRACTIONAL ORDER SWIFT-HOHENBERG EQUATIONS

Example 1

$$\frac{\partial^{\alpha}P(\psi,\tau)}{\partial\tau^{\alpha}} = (b-1)P(\psi,\tau) - 2\frac{\partial^{2P(\psi,\tau)}}{\partial\psi^2} - \frac{\partial^3 P(\psi,\tau)}{\partial\psi^3}, \qquad \alpha \in (0,1]. \tag{1. 1}$$

Initial condition is $\mathrm{P}(\psi,0) = \mathrm{P}_0 = \mathrm{e}^{\psi}$.
Using the iterative scheme define by equation (14), we have

$$P_0(\psi,\tau) = P(\psi), \tag{1. 2}$$

$$P_1(\psi,\tau) = L_\tau^{-\alpha}\left((b-1)\,P_0(\psi,\tau) - 2\frac{\partial^2 P_0(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P_0(\psi,\tau)}{\partial\psi^4}\right), \tag{1. 3}$$

$$\vdots$$

$$P_r(\psi,\tau) = L_\tau^{-\alpha}\left((b-1)\,P_{r-1}(\psi,\tau) - 2\,\partial^2\frac{P_{r-1}(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P_{r-1}(\psi,\tau)}{\partial\psi^4}\right). \tag{1. 4}$$

As we have initial source

$$\mathrm{P}_0(\psi,\tau) = \mathrm{P}(\psi,0) = \mathrm{e}^{\psi},$$

$$P_1(\psi,\tau) = \frac{(b-4)e^{\psi}\tau^{\psi}}{\Gamma(\alpha+1)}. \tag{1. 5}$$

By using iterative scheme define as

$$P_r(\psi,\tau) = L_\tau^{-\alpha}\left((b-1)P_{r-1}(\psi,\tau) - 2\frac{\partial^2 P_{r-1}(\psi,\tau)}{\partial\psi^2} - \frac{\partial^3 P_{r-1}(\psi,\tau)}{\partial\psi^3}\right). \tag{1. 6}$$

For specific values of $r = 2,3,\cdots$

$$P_2(\psi,\tau) = (b-4)^2\, e^{\psi}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{1. 7}$$

$$P_3(\psi,\tau) = (b-4)^3 e^{\psi}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{1. 8}$$

$$\vdots$$

The NAAM solution will become as,

$$P(\psi,\tau) = P_0(\psi,\tau) + P_1(\psi,\tau) + P_2(\psi,\tau) + P_3(\psi,\tau) + \cdots \tag{1. 9}$$

By putting corresponding values we get

$$P(\psi,\tau) = e^{\psi} + \frac{(b-4)e^{\psi}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-4)^2 e^{\psi}\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(b-4)^3 e^{\psi}\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \tag{1. 10}$$

For $\alpha = 1$, the solution converges to

$$P(\psi,\tau) = e^{\psi} + \frac{(b-4)e^{\psi}\tau}{\Gamma(2)} + \frac{(b-4)^2 e^{\psi}\tau^2}{\Gamma(3)} + \frac{(b-4)^3 e^{\psi}\tau^3}{\Gamma(4)} + \cdots \tag{1. 11}$$

$$P(\psi,\tau) = \sum_0^r \frac{(b-4)^r e^{\psi}\tau^r}{\Gamma(r+1)}. \tag{1. 12}$$

The exact solution is

$$P(\psi,\tau) = e^{\psi+(b-4)\tau}. \tag{1. 13}$$

TABLE I. COMPARISON OF NAAM AND ATHPM WITH EXACT SOLUTION

ψ	**NAAM ($\tau = 0.1$)**	**ATHPM ($\tau = 0.1$)**	**Exact ($\tau = 0.1$)**
0	0.6703200461	0.6703200305	0.6703200460
0.1	0.7408182207	0.7408182034	0.7408182207
0.2	0.8187307530	0.8187307338	0.8187307531
0.3	0.9048374184	0.9048373972	0.9048374180
0.4	1.000000000	0.9999999770	1.000000000
0.5	1.105170918	1.105170892	1.105170918
0.6	1.221402758	1.221402730	1.221402758
0.7	1.349858807	1.349858776	1.349858808
0.8	1.491824697	1.491824663	1.491824698
0.9	1.648721271	1.648721232	1.648721271
1.0	1.822118800	1.822118756	1.822118800

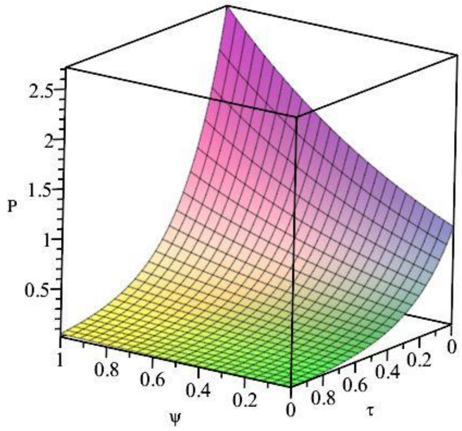


Figure 1.2. Exact Solution

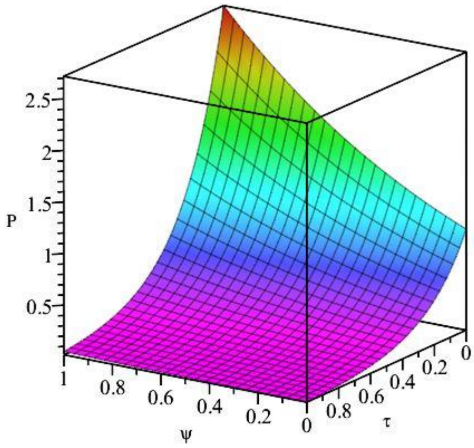


Figure 1.1. NAAM Solutions

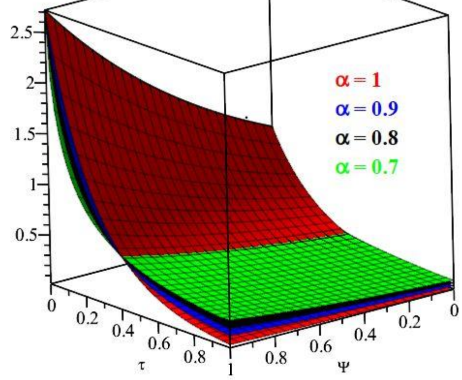


Figure 1.3. 3D Solutiion with $\alpha = 1, 0.9, 0.8, 0.7$

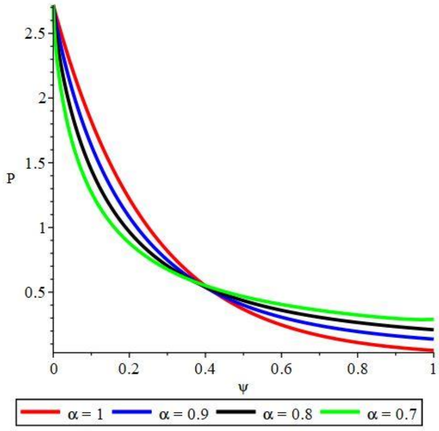


Figure 1.4. 2D Solutiion with $\alpha = 1, 0.9, 0.8, 0.7$

These plots present the approximate solutions to the problem using the independent variables ψ and τ, providing validation for the proposed approach through graphical representation. They also include 2D and 3D spectral analyses for fractional-order variables, highlighting multi-order solutions that closely match the experimental data.

Example 2

$$\frac{\partial^\alpha P(\psi,\tau)}{\partial\tau^\alpha} = (b-1)P(\psi,\tau) - 2\frac{\partial^2 P(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P(\psi,\tau)}{\partial\psi^4}, \quad \alpha \in (0,1]. \tag{2. 1}$$

Initial condition is

$$P(\psi,0) = P_0 = sin(\psi).$$

Using the iterative scheme define by equation (14), we have

$$P_0(\psi,\tau) = P(\psi), \tag{2. 2}$$

$$P_1(\psi,\tau) = L_\tau^{-\alpha}\left((b-1)P_0(\psi,\tau) - 2\frac{\partial^2 P_0(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P_0(\psi,\tau)}{\partial\psi^4}\right), \tag{2. 3}$$

$$\vdots$$

$$P_r(\psi,\tau) = L_\tau^{-\alpha}\left((b-1)P_{r-1}(\psi,\tau) - 2\frac{\partial^2 P_{r-1}(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P_{r-1}(\psi,\tau)}{\partial\psi^4}\right), \tag{2. 4}$$

$$P_0(\psi,\tau) = P(\psi,0) = sin(\psi), \tag{2. 5}$$

$$P_1(\psi,\tau) = \frac{((b-1)+1)sin(\psi)\tau^\psi}{\Gamma(\alpha+1)}. \tag{2. 6}$$

By using iterative scheme define as

$$P_r(\psi,\tau) = L_\tau^{-\alpha}\left((b-1)P_{r-1}(\psi,\tau) - 2\frac{\partial^2 P_{r-1}(\psi,\tau)}{\partial\psi^2} - \frac{\partial^3 P_{r-1}(\psi,\tau)}{\partial\psi^3}\right). \tag{2. 7}$$

For specific values of $r = 2,3,\cdots$

$$P_2(\psi,\tau) = \frac{((b-1)^2 + (b-1)+1)sin(\psi)\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{2. 8}$$

$$P_3(\psi,\tau) = \frac{((b-4)^3+(b-1)^2+(b-1)+1)sin(\psi)\tau^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{2. 9}$$

$$\vdots$$

The NAAM solution will become as

$$P(\psi,\tau) = P_0(\psi,\tau) + P_1(\psi,\tau) + P_2(\psi,\tau) + P_3(\psi,\tau) + \cdots \tag{2. 10}$$

By putting corresponding values we get

$$P(\psi,\tau) = sin(\psi) + \frac{((b-1)+1)sin(\psi)\tau^\psi}{\Gamma(\alpha+1)} + \frac{((b-1)^2 + (b-1)+1)sin(\psi)\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{((b-4)^3+(b-1)^2+(b-1)+1)sin(\psi)\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \tag{2. 11}$$

For $\alpha = 1$, the solution converges to:

$$P(\psi,\tau) = sin(\psi) + \frac{((b-1)+1)sin(\psi)\tau^\psi}{\Gamma(1+1)} + \frac{((b-1)^2+(b-1+1))sin(\psi)\tau^2}{\Gamma(2+1)} + \frac{((b-4)^3+(b-1)^2+(b-1)+1)sin(\psi)\tau^3}{\Gamma(3+1)} + \cdots \tag{2. 12}$$

$$P(\psi,\tau) = \sum_0^r \frac{((b-1)^r+(b-1)^{r-1}+1)sin(\psi)\tau^r}{\Gamma(r+1)}. \tag{2. 13}$$

The exact solution is

$$(\psi,\tau) = -\frac{1}{b}\, sin(\psi) \lim_{n\to\infty} \sum_{n=0}^{N} \frac{(1-(b-1)^{n+1})1}{\Gamma(n\alpha+1)}\, \tau^{n\alpha}. \tag{2. 14}$$

TABLE II. COMPARISON OF NAAM AND EXACT SOLUTION

ψ	**NAAM** $(\tau = 0.02)$	**Exact** $(\tau = 0.02)$	**NAAM** $(\tau = 0.1)$	**Exact** $(\tau = 0.1)$
0	1.799124094	1.799124093	1.797981356	1.797981356
0.1	1.870268913	1.870268913	1.797938290	1.797938291
0.2	1.061738456	1.061738459	1.798268491	1.798268491
0.3	1.183794955	1.183794959	1.892201837	1.892201839
0.4	1.799124087	1.799124087	1.612028341	1.612028342
0.5	1.917348471	1.917348473	1.789625489	1.789625488
0.6	1.947200111	1.947200112	1.797981356	1.797981356
0.7	1.873346001	1.873346000	1.789795135	1.789795136
0.8	0.999102833	1.000000123	1.896108112	1.896108112
0.9	1.799271108	1.799271108	1.797985383	1.797985383
1.0	1.910026474	1.910026472	1.782341103	1.782341104

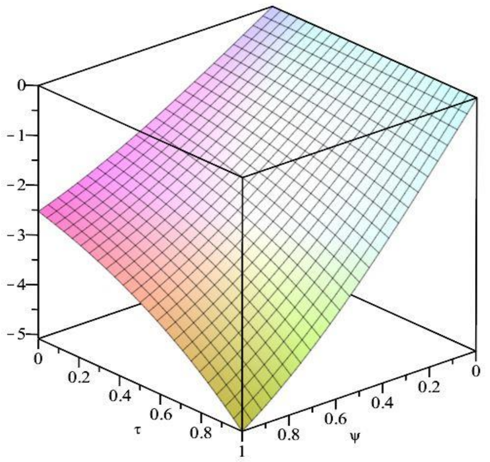


Figure 2.1. NAAM Solution

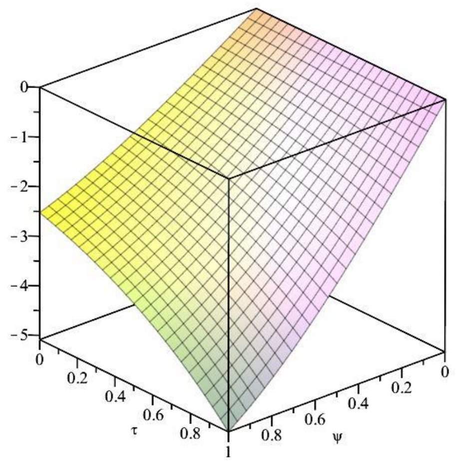


Figure 2.2. Exact Solution

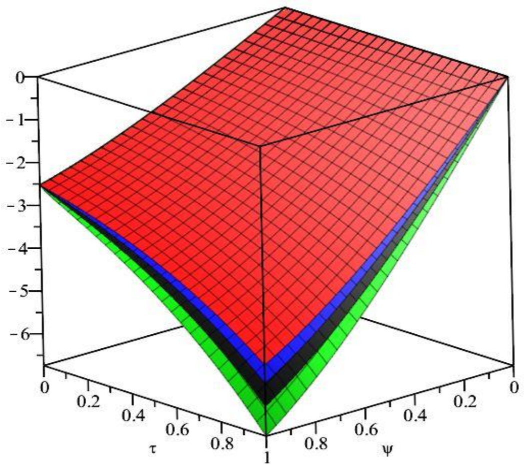


Figure 2.3. 3D Solutiion with $\alpha = 1, 0.9, 0.8, 0.7$

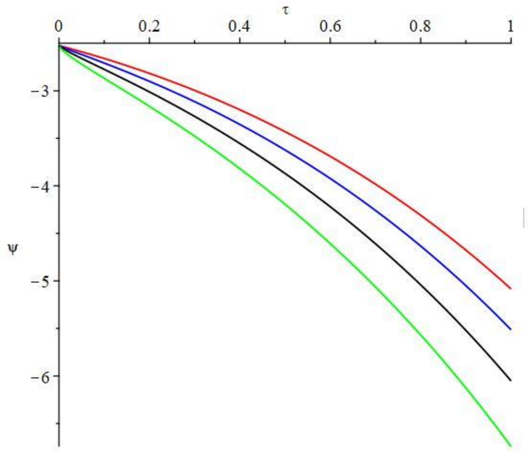


Figure 2.4. 2D Solutiion with $\alpha = 1, 0.9, 0.8, 0.7$

Here the plots showcase approximate solutions using the variables ψ and τ, validating the proposed approach through visual analysis. They further feature 2D and 3D spectral analyses of fractional-order variables, demonstrating multi-order solutions aligned with experimental data.

II. SOLUTION OF NONLINEAR FRACTIONAL ORDER SWIFT-HOHENBERG EQUATIONS

Example 3

$$\frac{\partial^\alpha P(\psi,\tau)}{\partial\tau^\alpha} = (b-1)W(\psi,\tau) - 2\frac{\partial^2 P(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P(\psi,\tau)}{\partial\psi^4} - \left(\frac{\partial P(\psi,\tau)}{\partial\psi}\right)^2 + P^2(\psi,\tau), \quad \alpha \in (0,1]. \tag{3. 1}$$

Initial condition is $\mathrm{P}(\psi, 0) = \mathrm{P}_0 = \mathrm{e}^{\psi}$.

Here $\mathrm{P}^2(\psi,\tau)$ and $\left(\frac{\partial \mathrm{P}(\psi,\tau)}{\partial\psi}\right)^2$ are nonlinear terms.

Using the iterative scheme define by equation (14), we have

$$P_0(\psi,\tau) = P(\psi), \tag{3. 2}$$

$$P_1(\psi,\tau) = L_\tau^{-\alpha}\left((b-1)P_0(\psi,\tau) - \frac{\partial^2 P_0(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P_0(\psi,\tau)}{\partial\psi^4}\right) + N_\tau^{-\alpha}, \tag{3. 3}$$

$$\vdots$$

$$P_r(\psi,\tau) = L_\tau^{-\alpha}\left((b-1)P_{\{r-1\}}(\psi,\tau) - 2\frac{\partial^2 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^4}\right) + P_{\tau_{\{r-1\}}}^{-\alpha}, \tag{3. 4}$$

Where $\mathrm{P}_{\tau_0}^{-\alpha} = \mathrm{P}_0^2(\psi,\tau) - \left(\frac{\partial \mathrm{P}_0(\psi,\tau)}{\partial\psi}\right)^2$, $\mathrm{P}_{\tau_{\{r-1\}}}^{-\alpha} = \mathrm{P}_{\{r-1\}}^2(\psi,\tau) - \left(\frac{\partial \mathrm{P}_{\{r-1\}}(\psi,\tau)}{\partial\psi}\right)^2$ are non-linear terms.

$$P_0(\psi,\tau) = P(\psi,0) = e^{\psi}, \tag{3. 5}$$

$$P_1(\psi,\tau) = \frac{(b-4)e^{\psi}\tau^\alpha}{\Gamma(\alpha+1)}. \tag{3. 6}$$

By using iterative scheme define as:

$$P_r(\psi,\tau) = L_\tau^{-\alpha}\Big((b-1)P_{\{r-1\}}(\psi,\tau) - \frac{2\partial^2 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^2} - \frac{\partial^4 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^4}\Big) + P_{\tau_{\{r-1\}}}^{-\alpha}, \quad (3.\ 7)$$

For specific values of $r = 2,3,\cdots$

$$P_2(\psi,\tau) = \frac{(b-4)^2 e^{\psi}\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (3.\ 8)$$

$$P_3(\psi,\tau) = \frac{(b-4)^3 e^{\psi}\tau^{3\alpha}}{\Gamma(3\alpha+1)}, \quad (3.\ 9)$$

$$\vdots$$

The NAAM solution will become as,

$$P(\psi,\tau) = P_0(\psi,\tau) + P_1(\psi,\tau) + P_2(\psi,\tau) + P_3(\psi,\tau) + \cdots \quad (3.\ 10)$$

By putting corresponding values we get:

$$P(\psi,\tau) = e^{\psi} + \frac{(b-4)e^{\psi}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-4)^2 e^{\psi}\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(b-4)^3 e^{\psi}\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \quad (3.\ 11)$$

For $\alpha = 1$, the solution converges to:

$$P(\psi,\tau) = e^{\psi} + \frac{(b-4)e^{\psi}\tau}{\Gamma(2)} + \frac{(b-4)^2 e^{\psi}\tau^2}{\Gamma(3)} + \frac{(b-4)^3 e^{\psi}\tau^3}{\Gamma(4)} + \cdots \quad (3.\ 12)$$

$$P(\psi,\tau) = \sum_0^r \frac{(b-4)^r e^{\psi}\tau^r}{\Gamma(r+1)} \quad (3.\ 13)$$

The exact solution is,

$$P(\psi,\tau) = e^{\psi+(b-4)\tau} \quad (3.\ 14)$$

TABLE III. COMPARISON OF NAAM AND EXACT SOLUTION

$\boldsymbol{\psi}$	**NAAM** ($\boldsymbol{\tau = 0.02}$)	**Exact** ($\boldsymbol{\tau = 0.02}$)	**NAAM** ($\boldsymbol{\tau = 0.1}$)	**Exact** ($\boldsymbol{\tau = 0.1}$)
0	1.799124094	1.799124093	1.797981356	1.797981356
0.1	1.870268913	1.870268913	1.797938290	1.797938291
0.2	1.061738456	1.061738459	1.798268491	1.798268491
0.3	1.183794955	1.183794959	1.892201837	1.892201839
0.4	1.799124087	1.799124087	1.612028341	1.612028342
0.5	1.917348471	1.917348473	1.789625489	1.789625488
0.6	1.947200111	1.947200112	1.797981356	1.797981356
0.7	1.873346001	1.873346000	1.789795135	1.789795136
0.8	0.999102833	1.000000123	1.896108112	1.896108112
0.9	1.799271108	1.799271108	1.797985383	1.797985383
1.0	1.910026474	1.910026472	1.782341103	1.782341104

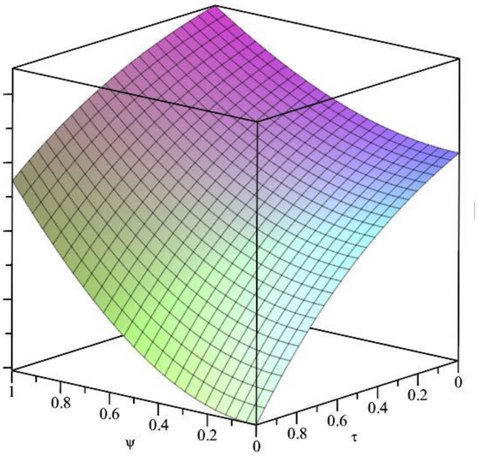


Figure 3.1. NAAM Solution

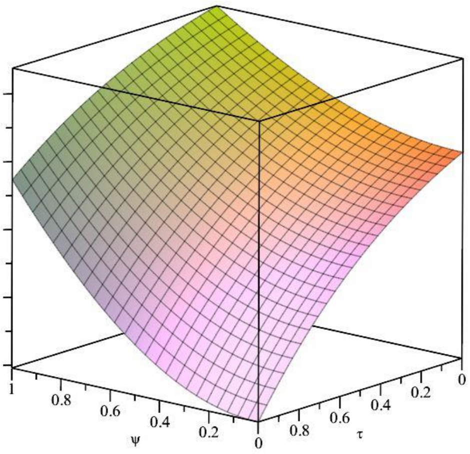


Figure 3.2. Exact Solution

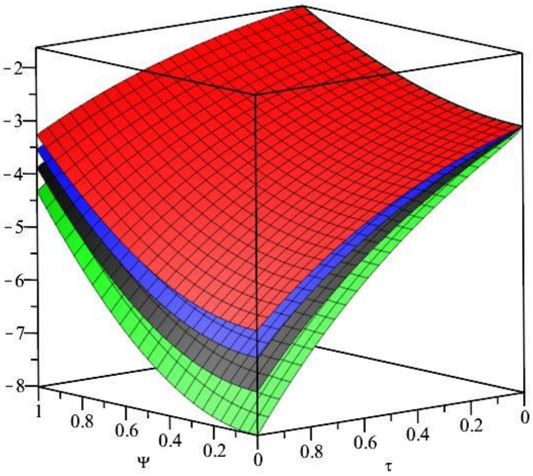


Figure 2.3. 3D Solutiion with $\alpha = 1, 0.9, 0.8, 0.7$

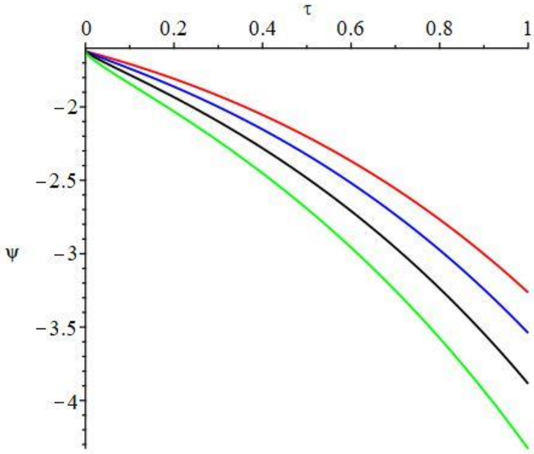


Figure 2.3. 2D Solutiion with $\alpha = 1, 0.9, 0.8, 0.7$

These plots illustrate approximate solutions to the problem using the independent variables ψ and τ, validating the proposed approach through graphical analysis. They also present spectral analysis in 2D and 3D for fractional-order variables, highlighting their capability to provide multi-order solutions that effectively align with experimental data.

Example 4

$$\frac{\partial^{\alpha}P(\psi,\tau)}{\partial\tau^{\alpha}} = (b-1)P(\psi,\tau) - \frac{2\partial^2 P(\psi,\tau)}{\partial\psi^2} + \sigma\frac{\partial^3 P(\psi,\tau)}{\partial\psi^3} - \frac{\partial^4 P(\psi,\tau)}{\partial\psi^4} - \left(\frac{\partial P(\psi,\tau)}{\partial\psi}\right)^2 + P^2(\psi,\tau), \alpha \in (0,1]. \qquad (4.\ 1)$$

Initial condition is $P(\psi,0) = P_0 = e^{\psi}$.

Here $P^2(\psi,\tau)$ and $\left(\frac{\partial P(\psi,\tau)}{\partial\psi}\right)^2$ are nonlinear terms.

Using the iterative scheme define by equation (14), we have

$$P_0(\psi,\tau) = P(\psi), \qquad (4.\ 2)$$

$$P_1(\psi,\tau) = L_{\tau}^{-\alpha}\left((b-1)P_0(\psi,\tau) - 2\frac{\partial^2 P_0(\psi,\tau)}{\partial\psi^2} + \sigma\frac{\partial^3 P_0(\psi,\tau)}{\partial\psi^3} - \frac{\partial^4 P_0(\psi,\tau)}{\partial\psi^4}\right) + N_{\tau}^{-\alpha}\left(P_0^2(\psi,\tau) - \left(\frac{\partial P_0(\psi,\tau)}{\partial\psi}\right)^2\right), \qquad (4.\ 3)$$

$$P_r(\psi,\tau) = L_{\tau}^{-\alpha}\left((b-1)P_{\{r-1\}}(\psi,\tau) - 2\frac{\partial^2 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^2} + \sigma\frac{\partial^3 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^3} - \frac{\partial^4 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^4}\right) + N_{\tau}^{-\alpha}\left(P_{\{r-1\}}^2(\psi,\tau) - \left(\frac{\partial P_{\{r-1\}}(\psi,\tau)}{\partial\psi}\right)^2\right), \qquad (4.\ 4)$$

where $P_0^2(\psi,\tau)$, $\left(\frac{\partial P_0(\psi,\tau)}{\partial\psi}\right)^2$ are non-linear terms.

$$P_0(\psi,\tau) = P(\psi,0) = e^{\psi} \qquad (4.\ 5)$$

$$P_1(\psi,\tau) = (b-4+\sigma)e^{\psi}\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}, \qquad (4.\ 6)$$

By using iterative scheme define as:

$$P_r(\psi,\tau) = L_{\tau}^{-\alpha}\left((b-1)P_{\{r-1\}}(\psi,\tau) - 2\frac{\partial^2 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^2} + \sigma\frac{\partial^3 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^3} - \frac{\partial^4 P_{\{r-1\}}(\psi,\tau)}{\partial\psi^4}\right) + N_{\tau}^{-\alpha}\left(P_{\{r-1\}}^2(\psi,\tau) - \left(\frac{\partial P_{\{r-1\}}(\psi,\tau)}{\partial\psi}\right)^2\right), \qquad (4.\ 7)$$

For specific values of $r = 2, 3, \cdots$

$$P_2(\psi,\tau) = (b-4+\sigma)^2 e^{\psi}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \qquad (4.\ 8)$$

$$P_3(\psi,\tau) = (b-4+\sigma)^3 e^{\psi}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} \qquad (4.\ 9)$$

$$\vdots$$

The NAAM solution will become as,

$$P(\psi,\tau) = P_0(\psi,\tau) + P_1(\psi,\tau) + P_2(\psi,\tau) + P_3(\psi,\tau) + \cdots \qquad (4.\ 10)$$

By putting corresponding values we get:

$$P(\psi,\tau) = e^{\psi} + (b-4+\sigma)e^{\psi}\frac{\tau^{\alpha}}{\Gamma(\alpha+1)} + (b-4+\sigma)^2 e^{\psi}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + (b-4+\sigma)^3 e^{\psi}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \qquad (4.\ 11)$$

For $\alpha = 1$, the solution converges to:

$$P(\psi,\tau) = e^{\psi} + (b-4+\sigma)e^{\psi}\frac{\tau}{\Gamma(2)} + (b-4+\sigma)^2 e^{\psi}\frac{\tau^2}{\Gamma(3)} + (b-4+\sigma)^3 e^{\psi}\frac{\tau^3}{\Gamma(4)} + \cdots \qquad (4.\ 12)$$

$$P(\psi,\tau) = \sum_0^r \frac{(b-4+\sigma)^r e^{\psi}\tau^r}{\Gamma(r+1)}. \qquad (4.\ 13)$$

The exact solution is,

$$P(\psi,\tau) = (b-4+\sigma)^n\, e^{\psi}\frac{1}{\Gamma(n\alpha+1)}\tau^{n\alpha} \qquad (4.\ 14)$$

TABLE IV. COMPARISON OF NAAM AND EXACT SOLUTION

ψ	**NAAM** $(\tau = 0.02)$	**Exact** $(\tau = 0.02)$	**NAAM** $(\tau = 0.1)$	**Exact** $(\tau = 0.1)$
0	1.799124094	1.799124093	1.797981356	1.797981356
0.1	1.870268913	1.870268913	1.797938290	1.797938291
0.2	1.061738456	1.061738459	1.798268491	1.798268491
0.3	1.183794955	1.183794959	1.892201837	1.892201839
0.4	1.799124087	1.799124087	1.612028341	1.612028342
0.5	1.917348471	1.917348473	1.789625489	1.789625488
0.6	1.947200111	1.947200112	1.797981356	1.797981356
0.7	1.873346001	1.873346000	1.789795135	1.789795136
0.8	0.999102833	1.000000123	1.896108112	1.896108112
0.9	1.799271108	1.799271108	1.797985383	1.797985383
1.0	1.910026474	1.910026472	1.782341103	1.782341104

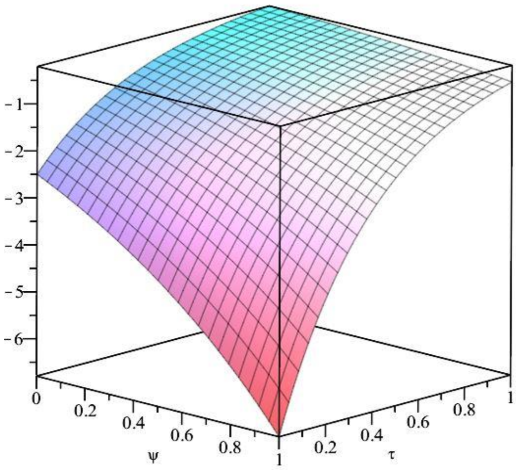
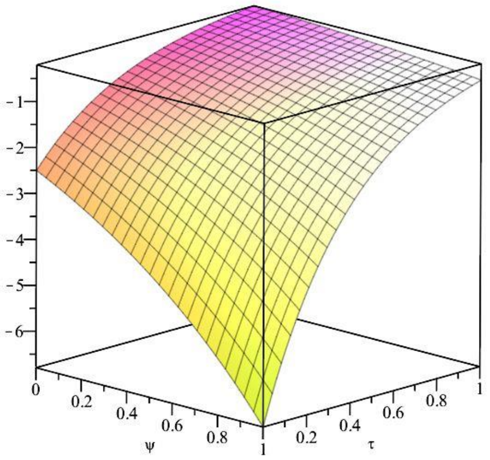


Figure 3.1. NAAM Solution



Fingure 3.2. Exact Solution

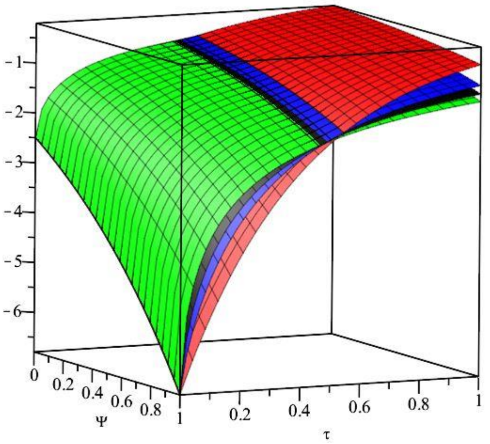


Figure 3.3. 3D Solutiion with $\alpha = 1, 0.9, 0.8, 0.7$

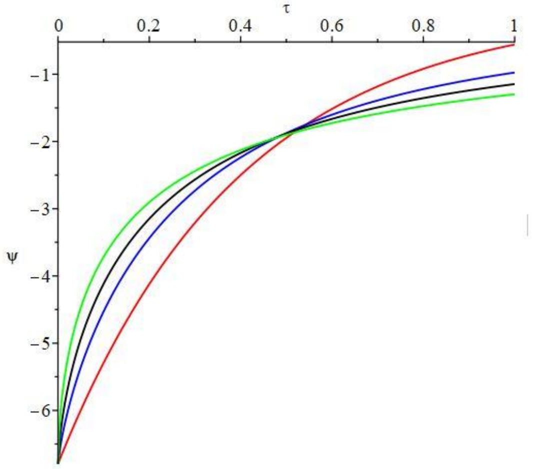


Figure 4.4. 2D Solutiion with $\alpha = 1, 0.9, 0.8, 0.7$

Here the plots depict approximate solutions to the problem, utilizing the independent variables ψ and τ, and substantiate the proposed methodology through comprehensive graphical analysis. Additionally, spectral analyses in 2D and 3D for fractional-order variables underscore their capacity to deliver multi-order solutions that align closely with experimental observations.

CONCLUSION

The New Approximate Analytical Method (NAAM) introduces an innovative and efficient framework for solving complex fractional-order equations. Its primary advantage lies in its ability to minimize computational requirements while ensuring high accuracy, making it a compelling alternative to conventional approaches. Unlike traditional methods, such as the Adomian Decomposition Method (ADM) or the Homotopy Perturbation Method (HPM), NAAM prioritizes simplicity and ease of implementation. This balance between user-friendliness and precision makes it particularly valuable for practical, real-world applications where efficiency is crucial.

Moreover, NAAM's versatility allows it to adapt seamlessly to a wide variety of equations, showcasing its ability to address diverse mathematical challenges. Its capability to offer precise solutions with less complexity positions it as a preferred choice for researchers and practitioners alike. By combining adaptability, accuracy, and accessibility, NAAM not only advances the field of fractional calculus but also provides a powerful tool for tackling intricate problems across multiple domains.

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In addition to his academic pursuits, Aslam has substantial teaching and leadership experience, having worked as a lecturer at various institutions and taken on administrative roles. His passion for mathematics and data-driven technologies fuels his ambition to further his studies with a Ph.D. in Mathematics, focusing on advanced research and academic development.

With a strong foundation in mathematical modeling, data analysis, and AI applications, Aslam is committed to contributing to both academia and industry through research, innovation, and education.

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