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Efficient Computation of Correlated Random Ordinates of Multivariate Weibull Distribution

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***Abstract*—**Random numbers are frequently used in engineering, data sciences, cryptography, crypto-currency wallets, gaming, gambling and various other fields. The desired characteristics of random numbers are uniformity of coverage and independence. Pseudo random number generators are used for this purpose, which are rated on the basis of long period between repeats and efficiency of algorithm. In many studies, sequences or arrays of random numbers from distributions of interest are required with the condition that these arrays have a given correlation structure. This includes but is not limited to the study of failure times of engines, landing gears or wings fitted on an airplane. Other areas of applications include Artificial Intelligence, Cryptographic Encryption, Economics, Health Sciences, Agriculture and the like. This study focused on efficient generation of bivariate and multivariate ordinates from Weibull distributions. Analytical as well as empirical approaches were employed and the generated random ordinates were checked for their fitness for use through chi-square goodness of fit test and comparison of the estimated parameter with the given values.

***Keywords*—** Correlated Ordinates, Bivariate and Multivariate Probability Distributions, Theoretical Values, Empirical Values.

I. Introduction

The bivariate Weibull distribution is a flexible tool that can be used in statistical modeling and reliability analysis to describe the combined behavior of two related random variables. The bivariate variant of the Weibull distribution, which is derived from the univariate variant, provides a strong framework for modeling dependencies between two variables. This makes it especially applicable in domains like risk assessment, survival analysis, and reliability engineering. Comprehending the joint distribution of two variables is important because it can help identify intricate relationships that might exist between them. In many real-world situations, the interplay of several components, each of which adds to the overall reliability, affects a system's performance or failure. To model such dependencies and obtain insights into the joint behavior of these variables, researchers and practitioners can utilize the mathematical representation offered by the bivariate Weibull distribution.

With its inherent flexibility, the Weibull distribution has garnered greater attention in the literature. The pdf and cdf of the univariate Weibull distribution are as follows, respectively.

A. Introduction to Univariate Weibull Distribution

The two-parameter Univariate Weibull Distribution pdf can be written as:

$$f(x) = \left(\frac{\gamma}{\beta}\right)\left(\frac{x}{\beta}\right)^{\gamma-1} e^{-\left(\frac{x}{\beta}\right)^{\gamma}};\ \gamma > 0,\ \beta > 0,\ x > 0 \qquad (1.1)$$

Where β and γ are the scale and shape parameters respectively
The corresponding cumulative distribution function is given by:

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\gamma}};\ \gamma > 0,\ \beta > 0,\ x > 0 \qquad (1.2)$$

This above relationship can be rewritten as:

$$x = \beta\, ln(-\ln(1-u))^{\frac{1}{\gamma}} \qquad (1.3)$$

B. Introduction to Bivariate Weibull Distribution

One formulation of the Survival Function of Bivariate Weibull Distribution suggested by Lee and Ven is:

$$F(x,y) = exp\left[-\left[\left(\frac{x}{\beta_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{y}{\beta_2}\right)^{\frac{\gamma_2}{\alpha}}\right]^{\alpha}\right];\ 0 < \alpha \le 1, \beta_1, \beta_2, \gamma_1, \gamma_2, x, y > 0. \qquad (1.4)$$

Note that x and y are independent for $\alpha = 1$.
Partial derivatives of the above survival function with respect to x and y leads to bivariate Weibull density function as follows:

$$f(x,y) = \frac{\delta^2}{\delta x\, \delta y}\bar{F} = \frac{\delta^2}{\delta x\, \delta y}\left\{exp\left[-\left[\left(\frac{x}{\beta_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{y}{\beta_2}\right)^{\frac{\gamma_2}{\alpha}}\right]^{\alpha}\right]\right\}$$

$$= \frac{\gamma_1}{\beta_1}\frac{\gamma_2}{\beta_2}\left(\frac{x}{\beta_1}\right)^{\frac{\gamma_1}{\alpha}-1}\left(\frac{y}{\beta_2}\right)^{\frac{\gamma_2}{\alpha}-1} exp\left[-\left[\left(\frac{x}{\beta_1}\right)^{\frac{\gamma_1}{\alpha}} + \left(\frac{y}{\beta_2}\right)^{\frac{\gamma_2}{\alpha}}\right]^{\alpha}\right]$$

$$\times$$

$$\left\{\left[\left(\frac{x}{\beta_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{y}{\beta_2}\right)^{\frac{\gamma_2}{\alpha}}\right]^{2\alpha-2}-\frac{(\alpha-1)}{\alpha}\left[\left(\frac{x}{\beta_1}\right)^{\frac{\gamma_1}{\alpha}}+\left(\frac{y}{\beta_2}\right)^{\frac{\gamma_2}{\alpha}}\right]^{\alpha-2}\right\} \quad (1.5)$$

II. LTERATURE REVIEW

Bivariate Weibull are specifically oriented towards applications in economics, finance and risk management as stated by Galiani [1]. Flores [2] used Weibull marginal to construct bivariate Weibull distributions. Kundu and Gupta [3] introduced the Marshall–Olkin bivariate Weibull distribution. It has been noted that lifetime data can be analyzed in one dimension fairly well using the inverse Weibull (IW) distribution, stated by [4]. He also discussed the different properties of Bivariate Weibull Distribution and its estimation[4]. Similarly, D. K. Al-Mutairi introduced the bivariate and multivariate distributions with weighted Weibull distribution [5]. D. Kundu introduced the Bayesian inference of unknown parameters of the progressively censored Weibull distribution [6]. A thorough discussion was held regarding the Weibull distribution's estimation and ideal progressive censoring schemes [7]. MC. Jones introduced a power generalized Weibull Distribution as a flexible parametric model for survival analysis [8]. Marcos Vinicius did a simulation study and introduced a bivariate modified Weibull distribution which was derived from Farlie-Gumbel-Morgenstern Copula [9]. Moreover, Weibull regression models with censored data have been discussed in [10]

III. METHODOLOGY TO GENERATE BIVARIATE WEIBULL ORDINATES

In order to compute the bivariate Weibull ordinates, the following methodology can be adopted:

Step 1: Generate pairs of uniformly distributed independent random number (u_i, v_i), $i = 1, 2, \dots, n$. Here, n=100,000

Step 2: Compute x_i using $x_i = \beta_1 (-\ln u_i)^{\frac{1}{\gamma_1}}$; $i = 1, 2, \dots, n$.

Step 3: Estimate an acceptable (maximum) value of y using $y_{max} = \beta_2 (-\ln 0.0001)^{\frac{1}{\gamma_2}}$.

Step 4: For a large number m (say $m = 20000$), divide the interval $(0, y_{max})$ into m small segments. For this purpose, compute $\Delta y = \frac{y_{max}}{m}$ and set $sy_1 = 0, sy_j = sy_{j-1} + \Delta y$; $j = 2, \dots, m$.

Step 5: Repeat the following steps for $i = 1, 2, \dots, n$:

a. Compute empirical values of density function $f(x_i, sy_j)$; $j = 1,2, \dots, m$.

b. Employing numerical integration, compute empirical values of $F(x_i, sy_j)$; using Δy and $f(x_i, sy_j)$; $j = 1,2, \dots, m$.

c. Compute scaled CDF values using $SF(x_i, sy_j) = \frac{F(x_i, sy_j)}{F(x_i, sy_m)}$; $j = 1,2, \dots, m$.

d. For $j = 1,2, \dots, m$, choose and set the value of sy_j as the required i^{th} random ordinate y_i for which the scaled CDF $SF(x_i, sy_j)$ is closets to v_i.

IV. RESULTS AND DISCUSSION

Results of 100,000 data points generated through Python code are:

Figure 1 to 10 shows the Histogram of the Generated Bivariate ordinates for a Weibull Distribution with parameters $\alpha = 0.1$ $\beta_1 = \beta_2 = 2.0$, $\gamma_1 = \gamma_2 = 1.5$. The value of α will go on rise from 0.1 to 1.

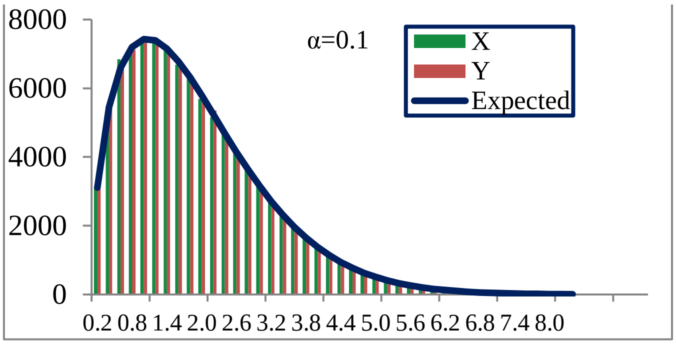

Figure 1: Histogram of X, Y and Expected Values

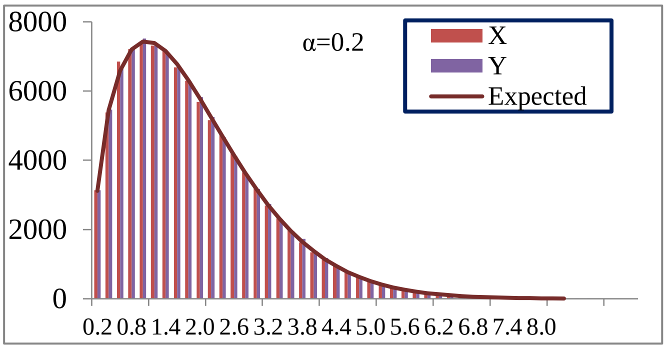

Figure 2: Histogram of X, Y and Expected Values

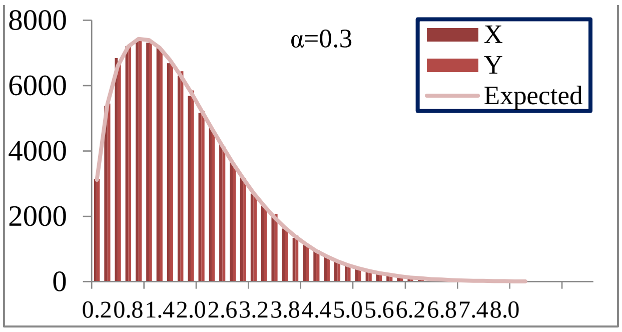

Figure 3: Histogram of X, Y and Expected Value

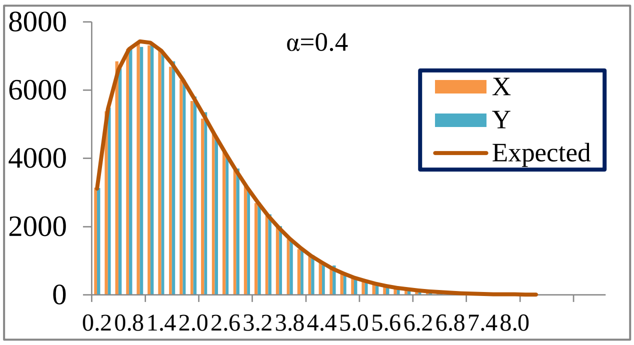

Figure 4: Histogram of X, Y and Expected Values

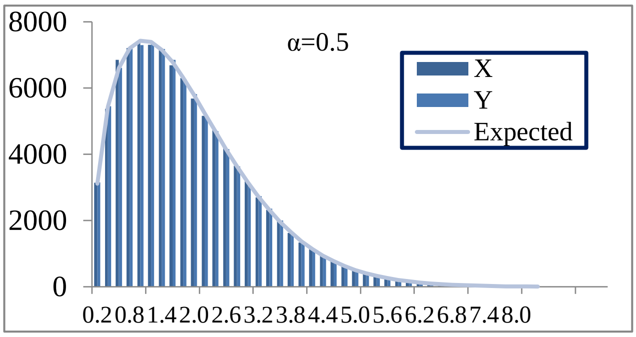

Figure 5: Histogram of X, Y and Expected Values

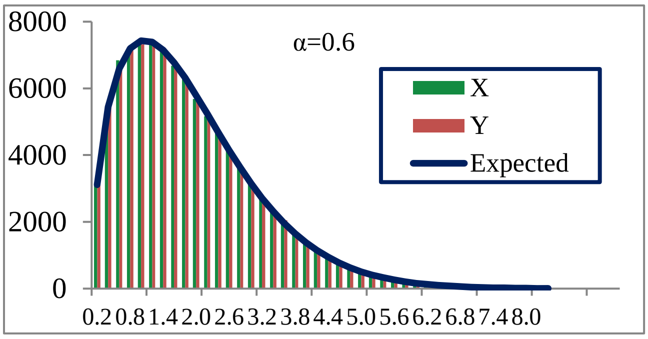

Figure 6: Histogram of X, Y and Expected Values

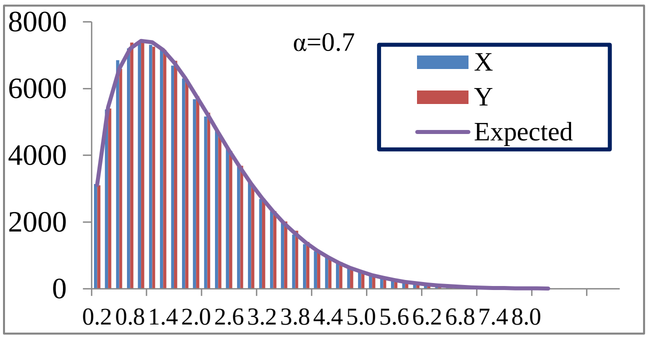

Figure 7: Histogram of X, Y and Expected Values

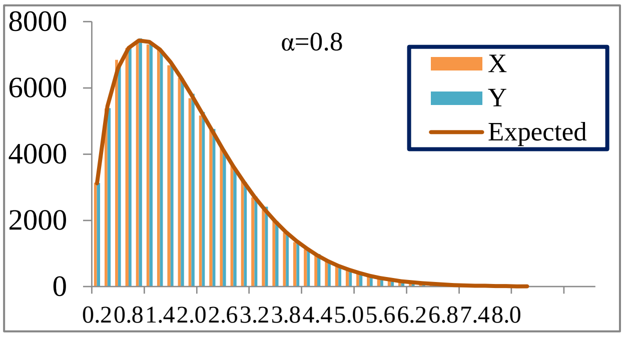

Figure 8: Histogram of X, Y and Expected Values

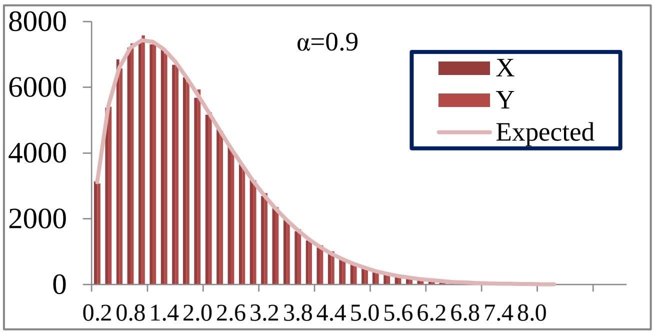

Figure 9: Histogram of X, Y and Expected Values

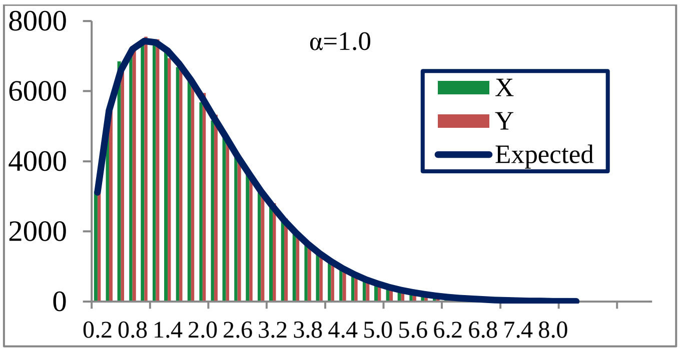

Figure 10: Histogram of X, Y and Expected Values

A. Comparison of Theoretical and Empirical parameters of Bivariate Weibull Ordinates

Furthermore, the comparison of theoretical values of parameters and moments with empirical values and its mathematical calculation are given below.

1) Theoretical parametrs are

$$\text{Theoretical } \mu = \beta \times \Gamma\left(1 + \tfrac{1}{\gamma}\right) \qquad (1.6)$$

$$\text{Theoretical } \sigma^2 = \beta^2 \left[\Gamma\left(1 + \tfrac{2}{\gamma}\right) - \left(\Gamma\left(1 + \tfrac{1}{\gamma}\right)\right)^2\right] \qquad (1.7)$$

For the case with scale and shape parameters $\beta = 2.0$ and $\gamma = 1.50$, the mean and variance are:

$$\mu = 2.0 \times \Gamma\left(1+\frac{1}{1.50}\right) = 2.0 + \Gamma\left(1+\frac{1}{1.50}\right) = 2 \times 0.9027 = 1.8054$$

$$\sigma^2 = 4\left[\Gamma\left(\frac{7}{3}\right) - \left(\Gamma\left(\frac{5}{3}\right)\right)^2\right] = 4[1.1906 - 0.8154] = 1.501$$

2) Empircal parameters are

The values of $\bar{x}$, $\bar{y}$, S_X^2 and S_Y^2 are the sample statistics computed from the descriptive statistics option of data analysis in Excel, given below in the table. Similarly, the empirical values calculated for the scale parameter $\beta^{\wedge}$ and shape parameter $\gamma^{\wedge}$ are calculated from the trendline of scatter plot of transformed generated values on x-axis and transformed theoretical values on y-axis.

Suppose the fitted equation (for data of x-ordinates) is as shown on the plot below.

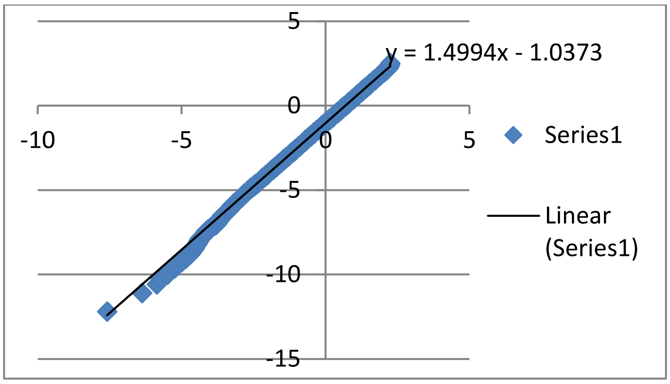

Figure 11: Showing the trendline equation of theoretical and empirical parameters for X (α=0.1)

The fitted equation here is $Y = 1.4994X - 1.0373$.
Here the Y-intercept is -1.0373 while the slope is 1.4994.
The slope of this equation is our estimate of gamma so $\hat{\gamma}_1 = 1.4994$. The estimate of the Beta is Y-intercept/ $\hat{\gamma}_1$ so $\hat{\beta}_1 = exp\left[\frac{-(-1.0373)}{1.4994}\right] = 1.997$

For computation of $\hat{\beta}_2$ and $\hat{\gamma}_2$, they are calculated from the trendline of scatter plot of transformed generated values on horizontal axis and transformed theoretical values on vertical axis. empirical parameters for Y (α=0.1).

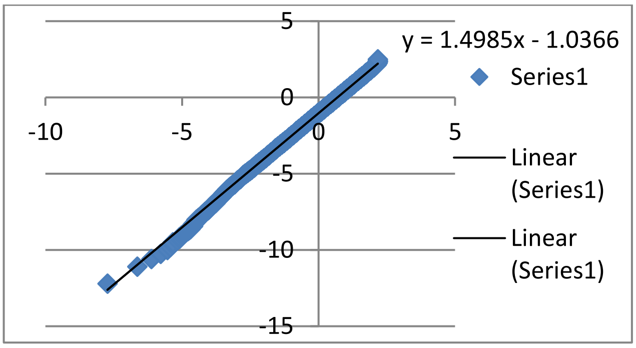

Figure 12: Showing the trendline equation of theoretical and Empirical parameters for Y (α=0.1)

Below is a table with the corresponding parameters that were estimated using the randomly generated ordinates. These closely match the values found in theory.

TABLE I. SHOWING THE THEORETCAL AND EMPIRICAL PARAMETERS OF BIVARATE WEIBULL DISTRBUTION

α	$\hat{\rho}$	$\bar{x}$	$\bar{y}$	S_X^2	S_Y^2	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
0.1	0.979	1.805	1.804	1.510	1.506	1.997	1.996	1.499	1.498
0.2	0.923	1.805	1.804	1.510	1.501	1.997	1.997	1.499	1.499
0.3	0.842	1.805	1.804	1.510	1.498	1.997	1.998	1.499	1.499
0.4	0.742	1.805	1.804	1.510	1.496	1.997	1.998	1.499	1.500
0.5	0.629	1.805	1.804	1.510	1.496	1.997	1.999	1.499	1.500
0.6	0.507	1.805	1.805	1.510	1.497	1.997	2.000	1.499	1.500
0.7	0.380	1.805	1.805	1.510	1.499	1.997	2.001	1.499	1.500
0.8	0.251	1.805	1.806	1.510	1.501	1.997	2.000	1.499	1.500
0.9	0.122	1.805	1.806	1.510	1.504	1.997	2.001	1.499	1.500
1.0	0.00	1.805	1.807	1.510	1.506	1.997	2.000	1.499	1.501

Similarly, the trendline for $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\rho}}$ is shown in the figure are calculated from the trendline of scatter plot of Rou on x-axis and alpha on y-axis.

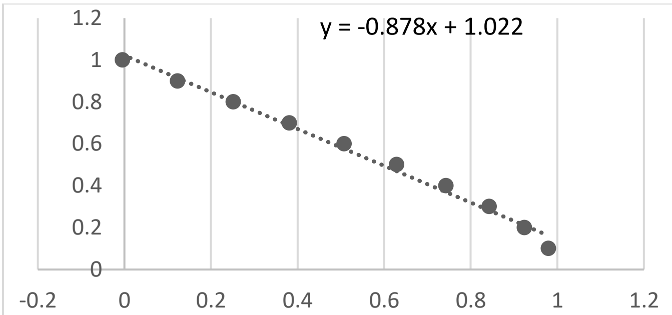

Figure 13: Showing the trendline equation between $\hat{\boldsymbol{\rho}}$ and $\boldsymbol{\alpha}$ for Weibull Ordinates

Suppose we need to generate Weibull distributed random ordinates for give values of $\hat{\beta}_1$ and $\hat{\gamma}_1$, $\hat{\beta}_2$ and $\hat{\gamma}_2$ with a correlation of 0.445.

We use the model to find α as below.

$$\alpha = 1.022 - 0.878 \times (0.445) = 0.63129$$

Similarly, from the above equation, correlation can also be found if the alpha is given.

B. Chi Square Goodness of Fit Test for X and Y

The Chi Square Goodness of fit test for both X and Y are given in the table below along with the Rou and Alpha values.

TABLE II. SHOWING THE CHI SQUARE AND P VALUE FOR BOTH X AND Y FROM (ALPHA=0.1 TO 1)

α	$\hat{\rho}$	χ_X^2	P Value	χ_Y^2	P Value
0.1	0.979	50.233	0.346	52.419	0.271
0.2	0.923	50.233	0.346	36.822	0.856
0.3	0.842	50.233	0.346	40.441	0.739
0.4	0.742	50.233	0.346	53.724	0.232
0.5	0.629	50.233	0.346	51.102	0.315
0.6	0.507	50.233	0.346	52.284	0.276
0.7	0.380	50.233	0.346	63.079	0.061
0.8	0.251	50.233	0.346	50.097	0.351
0.9	0.122	50.233	0.346	50.239	0.346
1.0	-0.00	50.233	0.346	72.976	0.008

*Null Hypothesis of Weibull Distribution ranging $0 < \alpha < 1$ is accepted as the critical value $\chi^2_{0.05,53} = 64.01$. It may be noted that for Weibull distribution, $0 < \alpha < 1$,we get the positive correlation, however, for α=1 or greater than we get the negative correlation as shown in the above table.

CONCUSLION

Hence, it is computed that the Random Ordinates which were selected follows the Bivariate Weibull Distribution because all of its theoretical and empirical parameters are same. Similarly, the goodness of fit statistic χ_X^2 and χ_X^2 are less than the critical value. Thus, it is demonstrated that the transformation of Uniform Random Ordinates U and V into a Bivariate Weibull Distribution substantiates the correctness of our Null Hypothesis for this distribution, confirming its acceptance.

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