Existence Criteria and Hyers-Ulam Theorem for a Coupled P-Laplacian System of Fractional Differential Equations

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Abstract — Dealing with high order coupled systems of FDEs through nonlinear p-Laplacian operator. We analyze existence, uniqueness & Hyer-Ulam stability (HUS) of the solutions by means of topological degree method. For this purpose, we transform the supposed problem into an integral system via Green’s function(s) and assume certain operator equivalent to the integral form of the problem. Then after, the results are proved with some necessary assumptions.

Keywords — Fractional differential equations (FDEs), Hyer-Ulam stability (HUS), topological degree theory, existence and uniqueness of solutions (EUS).

I. INTRODUCTION

The real world physical phenomena described by mathematical models of fractional differential equations (FDEs) are more constructive and practical in memory as compared to the models of integer order differential equations. Due to the application of FDEs, one can learn fractional calculus in diverse fields like metallurgy, signal and image processing, economics, fractal theory, biology and other disciplines [16-26]. Existence of solutions for FDEs is one of the most attracted research areas. For the different classes of FDEs, one can study different methods for existence and uniqueness of solutions. Various nonlinear mathematical models can be found in the scientific fields to study dynamic systems. The classical nonlinear operator \( \phi_p \) is one of the most important and frequently used nonlinear operators, which satisfies

\[
\frac{1}{p} + \frac{1}{q} = 1, \quad \phi_p(s) = |s|^{p-2}s, \quad p > 1 \text{ and } \phi_q(\theta) = \phi_p^{-1}(\theta).
\]

For details and applications of nonlinear operator \( \phi_p \), we pass on readers to [27-34].


Hu et al. [8] calculated existence of non-linear FDEs using the p-Laplacian operator:

\[
\begin{aligned}
D^\gamma_0 \left( \phi_p \left( D^\mu_0, \mu(t) \right) \right) + f \left( x, \mu(t), D^\mu_0, \mu(t) \right) &= 0, \quad t \in (0,1), \\
D^\delta_0, \mu(0) &= 0 = D^\delta_0, \mu(1),
\end{aligned}
\]

where 0 < \( \rho, \gamma < 1 \), 1 < \( \rho + \gamma < 2 \), \( D^\rho_0(0)^\gamma p, D^\rho_0(0)^\gamma \gamma \) are in the sense of Caputo derivatives.

Ali et al. [9] calculated the EUS and HUS for coupled system of FDEs:

\[
\begin{aligned}
D^\rho_0 x(t) &= f(t, y(t)), \quad t \in [0,1], \\
D^\rho_0 y(t) &= f(t, x(t)), \quad t \in [0,1], \\
x(0) &= 0, \quad x(t)|_{t=1} = \frac{1}{\Gamma(\rho)} \int_0^T (T-s)^{\rho-1} p(x(s)) ds, \\
y(0) &= 0, \quad y(t)|_{t=1} = \frac{1}{\Gamma(\rho)} \int_0^T (T-s)^{\rho-1} q(y(s)) ds,
\end{aligned}
\]

where \( \rho, \gamma, \sigma, \delta \in (1,2], D^\rho_0, D^\rho_0, \) are in the sense of Caputo derivatives, \( p, q \in L[0,1] \).

Khan et al. [36] recently calculated the existence and uniqueness of positive solutions and HUS for the following system of coupled FDEs:

\[
\begin{aligned}
D^\gamma_0 \left( \phi_p \left( D^\mu_0, x(t) \right) \right) &= -\Psi_1(t, y(t)), \quad D^\gamma_0 \left( \phi_p \left( D^\mu_0, y(t) \right) \right) = -\Psi_2(t, x(t)), \\
D^\delta_0 x(0) &= 0 = \left( \phi_p \left( D^\mu_0, x(t) \right) \right)|_{t=0} = D^\delta_0 x(t)|_{t=1}, \quad x(1) = \frac{\Gamma(2 - \delta_1) \phi_q \left( \phi_p \left( D^\mu_0, \psi(t, y(t)) \right) \right)|_{t=0}}{\eta_1^{1-\delta_1} \phi_q \left( \phi_p \left( D^\mu_0, \psi(t, y(t)) \right) \right)|_{t=1}},
\end{aligned}
\]

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\[ D_0^\gamma y(t) = 0 = \left( p_p \left( D_0^\delta y(t) \right) \right)' \bigg|_{t=0} = D_0^\delta y(t) \bigg|_{t=\eta_2}, y(1) = \frac{\Gamma(2 - \delta_2)}{\eta_2^2 - \delta_2} \varphi_0^x \phi_q \left( y_0^x \psi_2(t, x(t)) \right) \bigg|_{t=\eta_2} \]

where \( t \in [0,1], \rho, \gamma \in (1,2], \eta, \delta \in (0,1), \) for \( i = 1,2, \) and \( D_0^\gamma, D_0^\delta \) for \( i = 1,2 \) are denoting Caputo fractional derivatives.

In classical cases the fixed point theorems have some very strong conditions which limit the applications of the results to big extent for the study of many categories of FDEs and their coupled systems. Nowadays, degree theory plays a significant role in relaxing the necessary conditions required for the study of fixed points of operators and EUS for a large number of FDEs and its coupled systems for their solutions. Different types of degree theorems have been produced including the well-known Brouwer and Leray-Schauder theory which have been considered by a large number of scientists for the exploration of different aspects of fractional calculus especially dealing with existence of positive solution of differential equations involving integer order as well non-integer. A version of the degree theory acknowledged as topological degree which was introduced by Mawhin [10] and further expanded by Isiak [11] was considered for the existence results of linear as well nonlinear FDEs. The proposed technique is also known as prior estimation technique, which does not involve the compactness of operators. For new results on topological degree theory, we suggest the readers for the study of some recently developed results in [12-14].

Enthused from the abovementioned studies, we study the EUS and HUS of a coupled system with initial and boundary conditions and non-linear operator \( \phi_p \) using the topological degree method:

\[
\begin{aligned}
\left\{ \begin{array}{ll}
D_0^\alpha \left( p_p \left( D_0^\alpha u(t) \right) \right) = -\psi_1(t, v(t)), & D_0^\beta \left( p_p \left( D_0^\beta v(t) \right) \right) = -\psi_2(t, u(t)), \\
\left( p_p \left( D_0^\alpha u(t) \right) \right) |_{t=1} = 0, & \left( p_p \left( D_0^\alpha u(t) \right) \right) |_{t=0} = 0, \text{for } k = 1,2,3,\ldots, n - 1, \\
\left( p_p \left( D_0^\alpha v(t) \right) \right) |_{t=1} = 0, & \left( p_p \left( D_0^\alpha v(t) \right) \right) |_{t=0} = 0, \text{for } k = 1,2,3,\ldots, n - 1, \\
u^{(i)}(0) = 0, & \text{for } i = 0,1,2,\ldots, m - 2, m,\ldots, n - 1, u^{(m-1)}(1) = 0, \\
v^{(i)}(0) = 0, & \text{for } i = 0,1,2,\ldots, m - 2, m,\ldots, n - 1, v^{(m-1)}(1) = 0,
\end{array} \right.
\end{aligned}
\]

where \( \alpha, \beta_i \in (n - 1, n], \psi_1, \psi_2 \in L[0,1] \) and \( D_0^\alpha, D_0^\beta \) for \( i = 1,2 \) stand for Caputo fractional derivative \( \phi_p(N) = |N|^{\mu - 2}N \) is non-linear operator \( \phi_p \) satisfying \( 1/p + 1/q = 1, \phi_q \) represents inverse of \( \phi_p \). Here, we affirm that applying degree method to treat existence, uniqueness also to get the conditions for the stability of Hyers-Ulam to a coupled system of (FDEs) with \( \phi_p \) (1.1) has not been explored to our loyal awareness and understanding. Therefore, this work may get the attention of researchers to the study of Hyers-Ulam stability for more complex problems. We test enough conditions for the stability of EUS and HUS for system (1.1). Where \( [\alpha] \) is the integer part of \( \alpha \).

**Lemma 2.3.** Let \( \alpha \in (n - 1, n], \psi \in \mathcal{A} \mathcal{C} \mathcal{A} \mathcal{C}^{n-1}, \) then

\[
I_0^\alpha D_0^\alpha \psi(t) = \psi(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},
\]

For the \( c_i \in \mathbb{R} \) for \( i = 0,1,2,\ldots, n - 1. \)

Consider the space of real valued continuous functions \( V = \mathcal{C}([0,1], \mathbb{R}) \) with Topological norm \( \| v \| = \sup \{|v(t)|: 0 \leq t \leq 1 \} \) for \( v \in V \). The product space \( \omega = V \times V \) with the norm \( \| (v,w) \| = \| v \| + \| z \| \) is also Banach space. We give a notation \( S \) to the class of all bounded mappings in \( \omega \).

**Definition 2.4.** For the mapping \( \xi: S \to (0,\infty) \) Kuratowski measure of non-compactness is:

\[
\xi(M) = \inf \{ \rho > 0 : \exists \text{ finite cover for sets of diameter } \leq \rho \},
\]

where \( M \in S \).
Definition 2.5. Let $T: \emptyset \rightarrow V$ be bounded and continuous mapping with $\emptyset \subset V$. Then $T$ is an $\xi$-Lipschitz, where $\xi \geq 0$ if

$$\xi(T(M)) \leq \xi(M) \text{ for all bounded } M \subset \emptyset.$$  

And $T$ is a strict $\xi$-contraction with $\xi < 1$.  

Definition 2.6. The function $T$ is $\xi$-condensing if

$$\xi(T(M)) < \xi(M) \text{ for all bounded } M \subset \emptyset \text{ such that } \xi(M) > 0.$$  

Therefore $\xi(T(M)) \geq \xi(M)$ yields $\xi(M) = 0$.

More we have $T: \emptyset \rightarrow V$ is Lipschitz for $\xi > 0$, such that

$$\|T(v) - T(\bar{v})\| \leq \xi \|v - \bar{v}\| \text{ for all } v, \bar{v} \in \emptyset.$$  

The condition $\xi < 1$, then $T$ is a strict contraction. 

Proposition 2.7. The $T$ is said to be $\xi$-Lipschitz with $\xi = 0$ if and only if $T: \emptyset \rightarrow V$ is said to be compact.  

Proposition 2.8. The $T$ is said to be $\xi$-Lipschitz for constant value $\xi$ if and only if $T: \emptyset \rightarrow V$, is Lipschitz with constant $\xi$.

$$\begin{cases}
D_{0+}^{\beta_1} \left( \phi_p \left( D_{0+}^{\alpha_1} u(t) \right) \right) = -\psi_1(t, v(t)), \\
\left. \left( \phi_p \left( D_{0+}^{\alpha_1} u(t) \right) \right) \right|_{t=1} = 0, \\
\left. \left( \phi_p \left( D_{0+}^{\alpha_1} u(t) \right) \right) \right|_{t=0} = 0, \text{ for } k = 1, 2, 3, ..., n - 1, \\
u^{(0)}(0) = 0, \text{ for } i = 0, 1, 2, ..., m - 2, m, ..., n - 1, u^{(m-1)}(1) = 0,
\end{cases}$$

is given by the integral equation

$$u(t) = \int_0^1 H_{\beta_1}^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H_{\beta_1}^{\alpha_1}(s, t) \psi_1(t, v(t)) dt \right) ds,$$  

where $H_{\alpha_1}^{\beta_1}(t, s)$, $H_{\beta_1}^{\alpha_1}(t, s)$ are Green's function(s) defined by

$$H_{\alpha_1}^{\beta_1}(t, s) = \begin{cases}
\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} - t^{m-1} \frac{(1-s)^{\alpha_1-1}}{\Gamma(m) \Gamma(\alpha_1-m-1)}, & 0 \leq s \leq t \leq 1, \\
t^{m-1} \frac{(1-s)^{\alpha_1-1}}{\Gamma(m) \Gamma(\alpha_1-m-1)}, & 0 \leq t \leq s \leq 1,
\end{cases}$$

$$H_{\beta_1}^{\alpha_1}(t, s) = \begin{cases}
- \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} + \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leq s \leq t \leq 1, \\
\frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leq t \leq s \leq 1.
\end{cases}$$

Proof. Applying operator $I_{0+}^{\beta_1}$ on (3.1) and with help of Lemma (2.3), we proceed the following

$$\phi_p \left( D_{0+}^{\alpha_1} u(t) \right) = \psi_1(t, v(t)) + c_1 + c_2 t + \cdots + c_n t^{n-1}$$

Condition \label{eq3.5}

$$\left. \left( \phi_p \left( D_{0+}^{\alpha_1} u(t) \right) \right) \right|_{t=1} = 0, \text{ for } k = 1, 2, 3, ..., n - 1 \text{ results } c_2 = c_3 = \cdots = c_n = 0.$$  

And \label{eq3.5}

$$\left. \left( \phi_p \left( D_{0+}^{\alpha_1} u(t) \right) \right) \right|_{t=0} = 0,$$

implies

$$c_1 = \frac{1}{\Gamma(\beta_1)} \int_0^1 (t-s)^{\beta_1-1} \psi_1(s, v(s)) ds.$$

Theorem 2.9. \label{thm2.9} Let $T: V \rightarrow V$ be an $\xi$-condensing and $H = \{ z \in V: \text{ there exist } 0 \leq \lambda \leq 1 \text{ such that } z = \lambda Tz \}$. 

If $H$ is a bounded in $V$, there exists $a > 0$, and $H \subset M_a(0)$, with degree

$$deg(I - \lambda G, M_r(0), 0) = 1 \text{ for every } \lambda \in [0,1].$$

So, $T$ has at least one fixed point and collection of all fixed points of $T$ are contained in $M_a(0)$.

Lemma 2.10 \label{lem2.10} For the nonlinear operator $\phi_p$, we have

1. If $1 < p \leq 2, \ell_1 \ell_2 > 0$ and $\ell_1, |\ell_2| \geq m > 0$, then

$$|\phi_p(\ell_1) - \phi_p(\ell_2)| \leq (p - 1)m^{p-2}|\ell_1 - \ell_2|.$$  

2. If $p > 2$, $|\ell_1|, |\ell_2| \leq F$, then

$$|\phi_p(\ell_1) - \phi_p(\ell_2)| \leq (p - 1)F^{p-2}|\ell_1 - \ell_2|.$$  

III. MAIN RESULTS

Theorem 3.1. Let $\psi_1 \in C[0,1]$ be an integrable function satisfying (1.1). Then the solution of
By using values of $c_i$ for $i = 1, 2, \ldots, n$ and (3.5), we have
\[
\phi_p \left( \int_0^t \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1-1} \psi_1(s,v(s)) ds + \frac{1}{\Gamma(\beta_2)} \int_0^t (1-s)^{\beta_2-1} \psi_1(s,v(s)) ds \right)_{t=1} = \int_0^t H^{\beta_1}(t, s) \psi_1(s,v(s)) ds,
\] (3.7)
where $H^{\beta_1}(t, s)$ is a Green’s function given in (3.4).
Applying $\phi_q = \phi_p^{-1}$ in (3.7), we get
\[
\int_0^t H^{\beta_1}(t, s) \psi_1(s,v(s)) ds = \phi_q \left( \int_0^t H^{\beta_1}(t, s) \psi_1(s,v(s)) ds \right).
\] (3.8)
Applying operator $I_{0+}^{\alpha_1}$ on (3.8) and Lemma 2.10, we get
\[
\int_0^t H^{\beta_1}(t, s) \psi_1(s,v(s)) ds + z_1 + z_2 t + \ldots + z_m t^{m-1} + \ldots + z_n t^{n-1} \leq \int_0^t H^{\beta_2}(t, s) \psi_1(s,v(s)) ds.
\] (3.9)
Using condition $u(i)(0) = 0$ for $i = 0, 1, 2, \ldots, m-2, m, \ldots, n-1$, we obtain $z_1 = z_2 = \ldots = z_m = z_{m+1} = \ldots z_n = 0$.
From condition $u(1)(1) = 0$, we have
\[
2770 \leq \int_0^1 H^{\beta_1}(t, s) \psi_1(s,v(s)) ds \leq \int_0^1 H^{\beta_2}(t, s) \psi_1(s,v(s)) ds
\] (3.10)
Now putting the values of $z_i$ for $i = 1, 2, \ldots, m, \ldots, n$ in (3.9), we have
\[
\int_0^t H^{\beta_1}(t, s) \psi_1(s,v(s)) ds = \phi_q \left( \int_0^t H^{\beta_1}(t, s) \psi_1(s,v(s)) ds \right)_{t=1}
\]
\[
= \int_0^t H^{\beta_1}(t, s) \psi_1(s,v(s)) ds.
\] (3.11)
where $H^{\alpha_1}(t, s), H^{\beta_2}(t, s)$ are defined by (3.3), (3.4), respectively, as Green’s function(s).
Following Theorem 3.1, we may write our problem in the following system
\[
u(t) = \int_0^t H^{\alpha_2}(t, s) \psi_1(s,v(s)) ds,
\] (3.12)
\[
\int_0^t H^{\alpha_2}(t, s) \psi_1(s,v(s)) ds,
\] (3.13)
where $H^{\alpha_2}(t, s)$ are Green’s function(s) defined by
\[
H^{\alpha_2}(t, s) = \begin{cases} \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} - \frac{t^{m-1}}{\Gamma(m)\Gamma(\alpha_2 - (m-1))}(1-s)^{\alpha_2-m}, & 0 \leq s \leq t \leq 1, \\
-\frac{t^{m-1}}{\Gamma(m)\Gamma(\alpha_2 - (m-1))}(1-s)^{\alpha_2-m}, & 0 \leq t \leq s \leq 1, \end{cases}
\] (3.14)
\[
H^{\beta_2}(t, s) = \begin{cases} \frac{(t-s)^{\beta_2-1}}{\Gamma(\beta_2)} + \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}, & 0 \leq s \leq t \leq 1, \\
\frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}, & 0 \leq t \leq s \leq 1, \end{cases}
\] (3.15)
Define $T^*_i: V \rightarrow V \; (i = 1,2)$ by

$$T^*_1 u(t) = \int_0^1 H^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H^{\beta_1}(s, \varphi) \psi_1(\varphi, v(\varphi)) d\varphi \right) ds$$

$$T^*_2 v(t) = \int_0^1 H^{\alpha_2}(t, s) \phi_q \left( \int_0^1 H^{\beta_2}(s, \varphi) \psi_2(\varphi, u(\varphi)) d\varphi \right) ds$$

(3.16) (3.17)

We further define $F(u, v) = (T^*_1(u), T^*_2(v))$. Then, with help of Theorem 3.1, the solution of (3.12), (3.13) is equivalent to any fixed point, say $(x, y)$, of operator equation

$$(x, y) = F(x, y).$$

To proceed further, we need following assumptions in the main results of the paper.

(Q1) With positive constant values of $a_1, b_1, \mathbb{M}_{\psi_1}, \mathbb{M}_{\psi_2}$, and $k_1, k_2 \in [0,1]$, functions $\psi_1, \psi_2$ satisfy the following growth conditions

$$|\psi_1(x, v)| \leq \phi_p(a_1 |v|^{k_1} + \mathbb{M}_{\psi_1}),$$

$$|\psi_2(x, u)| \leq \phi_p(b_1 |u|^{k_2} + \mathbb{M}_{\psi_2}).$$

(Q2) There exist real valued constants $\lambda_{\psi_1}, \lambda_{\psi_2}$ such that for all $u, v, x, y \in V$,

$$|\psi_1(t, v) - \psi_1(t, x)| \leq \lambda_{\psi_1} |v - x|,$$

$$|\psi_2(t, u) - \psi_2(t, y)| \leq \lambda_{\psi_2} |u - y|.$$

For simplicity in calculations, we define the following terms:

$$Y_1 = \left( \frac{1}{\Gamma(\alpha_1 + 1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_1 + 1)} + \frac{1}{\Gamma(\beta_2 + 1)} \right),$$

$$\rho_1 = \frac{1}{\Gamma(\alpha_1 + 1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)} \left( \frac{1}{\Gamma(\beta_1 + 1)} + \frac{1}{\Gamma(\beta_2 + 1)} \right).$$

Theorem 3.2. With assumption (Q1), the operator $F: \omega^* \rightarrow \omega^*$ is continuous and satisfies following growth condition

$$F(u, v)(t) \leq \delta^* \| (u, v) \|^k + \rho_1^*.$$  

(3.19)

For each $(u, v) \in \mathcal{M}_r \subset \omega^*$

**Proof.** Consider a bounded set $\mathcal{M}_r = \{(u, v) \in \omega: \| (u, v) \| \leq r \}$ with sequence $\{(u_n, v_n)\}$ converging to $(u, v)$ in $\mathcal{M}_r$. To show that $\| T^*(u_n, v_n) - T^*(u, v) \| \to 0$ as $n \to \infty$. Let us consider

$$[T^*_1 u_n(t) - T^*_1 u(t)] = \int_0^1 H^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H^{\beta_1}(s, \varphi) \psi_1(\varphi, v_n(\varphi)) d\varphi \right) ds - \int_0^1 H^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H^{\beta_1}(s, \varphi) \psi_1(\varphi, v(\varphi)) d\varphi \right) ds$$

$$\leq \int_0^1 |H^{\alpha_1}(t, s)| \phi_q \left( \int_0^1 |H^{\beta_1}(s, \varphi)| \psi_1(\varphi, v_n(\varphi)) d\varphi \right) ds$$

$$- \phi_q \left( \int_0^1 |H^{\beta_1}(s, \varphi)| \psi_1(\varphi, v(\varphi)) d\varphi \right) ds,$$

(3.20)

And
\[ |T^*_2 v_n(t) - T^*_2 v(t)| \]

\[ = \left| \int_0^1 \left( \int_0^1 H^{a_2}(s, \varphi) \psi_2(\varphi, u_n(\varphi)) d\varphi \right) ds \right| \]

\[ - \left| \int_0^1 \left( \int_0^1 \left| H^{a_2}(s, \varphi) \psi_2(\varphi, u(\varphi)) \right| d\varphi \right) ds \right| \]

\[ \leq \left| \int_0^1 \left| H^{a_2}(s, \varphi) \right| \left( \int_0^1 \left| \psi_2(\varphi, u(\varphi)) \right| d\varphi \right) ds \right| \]

\[ - \left( \int_0^1 \left( \int_0^1 \left| \psi_2(\varphi, u(\varphi)) \right| d\varphi \right) ds \right) \]

\[ \leq \left( \frac{1}{\Gamma(\alpha_1 + 1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_1 + 1)} + \frac{1}{\Gamma(\beta_1 + 1)} \right) ^{q-1} (a_1 \parallel v \parallel k_1 + M_{\psi_1}^\ast) \]

\[ = Y_1(a_1 \parallel v \parallel k_1 + M_{\psi_1}^\ast). \]

And

\[ |T^*_2 v(t)| \]

\[ \leq \left( \frac{1}{\Gamma(\alpha_2 + 1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_2 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_2 + 1)} + \frac{1}{\Gamma(\beta_2 + 1)} \right) ^{q-1} (b_1 \parallel u \parallel k_2 + M_{\psi_2}^\ast) \]

\[ = Y_2(b_1 \parallel u \parallel k_2 + M_{\psi_2}^\ast). \]

With the help of (3.22), (3.23), we proceed

\[ |F^*(u, v)(t)| \leq Y_1(a_1 \parallel v \parallel k_1 + M_{\psi_1}^\ast) + Y_2(b_1 \parallel u \parallel k_2 + M_{\psi_2}^\ast) \]

\[ \leq \delta^* \parallel (u, v) \parallel \parallel v \parallel + \rho_1^* \]

This completes the proof.

**Theorem 3.3.** Assume that \((Q_1)\) holds true. Then \(F^*: \omega^* \to \omega^*\) is compact and \(\xi - \text{Lipschitz with constant zero}\).

**Proof.** Theorem 3.2 implies that \(F^*: \omega \to \omega\) is bounded. Next, let \(B \subset M_r \subset \omega^*\). Then, by \((Q_1)\), Lemma 3.1, Eq. (3.12), (3.13), then for any \(t_1, t_2 \in [0,1]\), we have
\[ |T_1^*u(t_1) - T_1^*u(t_2)| = \left| \int_0^1 H^{a_1}(t_1, s) \phi_q \left( \int_0^1 H^{b_1}(s, \varrho) \psi_1(\varrho, v(\varrho)) d\varrho \right) ds - \int_0^1 H^{a_2}(t_2, s) \phi_q \left( \int_0^1 H^{b_2}(s, \varrho) \psi_2(\varrho, u(\varrho)) d\varrho \right) ds \right| \]
\[
\leq \int_0^1 |H^{a_1}(t_1, s) - g^{a_1}(t_2, s)| \phi_q \left( \int_0^1 |H^{b_1}(s, \varrho)| \phi_p(b_1) |v|^k_1 + M_{\psi_1}^* \right) d\varrho \]
\[
\leq \left( \frac{|t_1^{a_1} - t_2^{a_1}|}{\Gamma(\alpha_1 + 1)} + \frac{|t_1^{m_1} - t_2^{m_1}|}{\Gamma(\alpha_1 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_1 + 1)} + \frac{1}{\Gamma(\beta_1 + 1)} \right)^{q-1} (a_1 \ |v|^k_1 + M_{\psi_1}^*), \tag{3.25} \]

\[ |T_2^*v(t_1) - T_2^*v(t_2)| = \left| \int_0^1 H^{a_2}(t_1, s) \phi_q \left( \int_0^1 H^{b_2}(s, \varrho) \psi_2(\varrho, u(\varrho)) d\varrho \right) ds - \int_0^1 H^{a_2}(t_2, s) \phi_q \left( \int_0^1 H^{b_2}(s, \varrho) \psi_2(\varrho, u(\varrho)) d\varrho \right) ds \right| \]
\[
\leq \int_0^1 |H^{a_2}(t_1, s) - H^{a_2}(t_2, s)| \phi_q \left( \int_0^1 |H^{b_2}(s, \varrho)| \phi_p(b_2) |u|^k_2 + M_{\psi_2}^* \right) d\varrho \]
\[
\leq \left( \frac{|t_1^{a_2} - t_2^{a_2}|}{\Gamma(\alpha_2 + 1)} + \frac{|t_1^{m_2} - t_2^{m_2}|}{\Gamma(\alpha_2 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_2 + 1)} + \frac{1}{\Gamma(\beta_2 + 1)} \right)^{q-1} b_1 |u|^k_2 + M_{\psi_2}^*. \tag{3.26} \]

From (3.25), (3.26), we have
\[ |F^*(u, v)(t_2) - F^*(u, v)(t_1)| \]
\[
\leq \left( \frac{|t_1^{a_1} - t_2^{a_1}|}{\Gamma(\alpha_1 + 1)} + \frac{|t_1^{m_1} - t_2^{m_1}|}{\Gamma(\alpha_1 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_1 + 1)} + \frac{1}{\Gamma(\beta_1 + 1)} \right)^{q-1} (a_1 \ |v|^k_1 + M_{\psi_1}^*) \
\]
\[
+ \left( \frac{|t_1^{a_2} - t_2^{a_2}|}{\Gamma(\alpha_2 + 1)} + \frac{|t_1^{m_2} - t_2^{m_2}|}{\Gamma(\alpha_2 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_2 + 1)} + \frac{1}{\Gamma(\beta_2 + 1)} \right)^{q-1} b_1 |u|^k_2 \
+ M_{\psi_2}^*. \tag{3.27} \]

As \( t_1 \to t_2 \), (3.27) approaches to zero which may be observe on the right side. Thus, the operator \( F^* = (T_1^*, T_2^*) \) is an equicontinuous on \( \mathcal{B} \). Arzela-Ascoli theorem implies that \( F^*(\mathcal{B}) \) is compact. Ultimately, \( \mathcal{B} \) is \( \xi - \) Lipschitz with constant zero.

**Theorem 3.4.** With assumptions \( Q_1 < Q_2 \ and \delta^* < 1 \), the system of FDEs with \( \phi_p \ (1.1) \) has a bounded solution in \( \omega^* \).

**Proof.** For existence of solution of the coupled differential system of fractional order (1.1), we take help from Theorem 2.9. Let us consider the set
\[ S = \{(u, v) \in \omega^*: \text{there exist } \lambda \in [0,1], \text{such that } (u, v) = \lambda F(u, v)\}, \]

We show that \( S \) is bounded. For this we assume a contrary path. Let for some \( (u, v) \in S \), such that \( \| (u, v) \| = J \to \infty \). But from Theorem 3.2, we have
\[
\| (u, v) \| = \| \lambda F(u, v) \| \leq \| F(u, v) \| \leq \delta^* \| (u, v) \| k + \rho^*_1. \tag{3.28} \]

Since \( \| (u, v) \| = J \), then (3.28) implies
\[
\| (u, v) \| \leq \delta^* \| (u, v) \| k + \rho^*_1 \]
\[
1 \leq \frac{\delta^*}{1 - \frac{1}{\gamma^*}} + \frac{\rho^*_1}{\gamma^*} \to 0 \text{ as } J \to \infty. \]
Recently, Khan et al. [35] studied the stability of Hyers-Ulam for the following system of FDEs with $\phi_p$: 

**Theorem 3.5.** Assume that $Q_1 - Q_2$ hold. Then Eq. (1.1) has a unique solution provided that $\vartheta_1 + \vartheta_2 < 1$.

**Proof.** From (3.16), (3.17) and assumptions $Q_1 - Q_2$ and Lemma (2.10). Then for any $t_1, t_2 \in [0,1]$, we proceed

$$
\| T_1^* u(t) - T_1^* \tilde{u}(t) \|
= \left| \int_0^1 H^{\alpha_1}(t, s) \psi_1(t, v(s)) ds - \int_0^1 H^{\alpha_1}(t, s) \psi_1(t, \tilde{v}(s)) ds \right|
= \left| \int_0^1 |H^{\alpha_1}(t, s)| \left| \psi_1(t, v(s)) - \psi_1(t, \tilde{v}(s)) \right| ds \right|
\leq (p - 1) \sigma^{p-2} \lambda \psi_1 \left( \frac{1}{\Gamma(\alpha_1 + 1)} + \frac{1}{\Gamma(m) \Gamma(\alpha_1 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_1 + 1)} + \frac{1}{\Gamma(\beta_1 + 1)} \right) \| v(t) - \tilde{v}(t) \|
= \vartheta_1 \| v(t) - \tilde{v}(t) \|
$$

And

$$
\| T_2^* v(t) - T_2^* \tilde{v}(t) \|
= \left| \int_0^1 H^{\alpha_2}(t, s) \psi_2(t, u(s)) ds - \int_0^1 H^{\alpha_2}(t, s) \psi_2(t, \tilde{u}(s)) ds \right|
= \left| \int_0^1 |H^{\alpha_2}(t, s)| \left| \psi_2(t, u(s)) - \psi_2(t, \tilde{u}(s)) \right| ds \right|
\leq (p - 1) \sigma^{p-2} \lambda \psi_2 \left( \frac{1}{\Gamma(\alpha_2 + 1)} + \frac{1}{\Gamma(m) \Gamma(\alpha_2 - m + 2)} \right) \left( \frac{1}{\Gamma(\beta_2 + 1)} + \frac{1}{\Gamma(\beta_2 + 1)} \right) \| u(t) - \tilde{u}(t) \|
= \vartheta_2 \| u(t) - \tilde{u}(t) \|
$$

From (3.29), (3.30), we have

$$
|F^*(u, v)(t) - F^*(\tilde{u}, \tilde{v})(t)| \leq \vartheta_1 \| v(t) - \tilde{v}(t) \| + \vartheta_2 \| u(t) - \tilde{u}(t) \|
\leq (\vartheta_1 + \vartheta_2) \| (u, v)(t) - (\tilde{u}, \tilde{v})(t) \|
$$

With assumption $\vartheta_1 + \vartheta_2 < 1$, Banach’s contraction principle implies that $F^*$ has a unique fixed point. As a result, the solution of the system of fractional order with $\phi_p$ (1.1) is unique.

Recently, Khan et al. [35] studied the stability of Hyers-Ulam for the following system of FDEs with $\phi_p$.
The system of integral equations given by (3.12), (3.13) are
\[
\begin{align*}
\left( D^\alpha_{0+} \left( \Phi_p \left( D^\alpha_{0+} u(t) \right) \right) + \psi_1(t, v(t)) \right) &= 0, \\
\left( D^\alpha_{0+} \left( \Phi_p \left( D^\alpha_{0+} v(t) \right) \right) + \psi_2(t, u(t)) \right) &= 0, \quad \text{(3.32)}
\end{align*}
\]
\[
\left( \Phi_p \left( D^\alpha_{0+} u(t) \right) \right) |_{t=1} = t^\beta_{0+} \left( \psi_1(t, v(t)) \right) |_{t=1},
\]
\[
\left( \Phi_p \left( D^\alpha_{0+} v(t) \right) \right) |_{t=1} = t^\beta_{0+} \left( \psi_2(t, u(t)) \right) |_{t=1},
\]
\[
\Phi_p \left( D^\alpha_{0+} u(t) \right) = \Phi_p \left( D^\alpha_{0+} v(t) \right) \quad \text{for } i = 0.
\]
\[
\begin{align*}
u(0) &= v(0), \\
u(1) &= 1, \\
u(0) &= 0 = \nu''(0), \\
u(1) &= 0,
\end{align*}
\]
Where \(2 < \alpha, \beta_1 < 3, \psi_1, \psi_2 \in L[0,1], \) and \( D^\alpha_{0+}, D^\beta_{0+} \) for \( i = 1, 2 \) are in Caputo sense.

They converted (3.32) system into following coupled system of Hammerstein-type integral equations given below:
\[
\begin{align*}
u(t) &= \int_0^1 H^\alpha_{0+} (t, s) \left( \int_0^1 H^\beta_{0+} (s, \theta) \psi_1 (\theta, v(\theta)) d\theta \right) ds, \quad \text{(3.33)} \\
\psi(t) &= \int_0^1 H^\alpha_{0+} (t, s) \left( \int_0^1 H^\beta_{0+} (s, \theta) \psi_2 (\theta, u(\theta)) d\theta \right) ds, \quad \text{(3.34)}
\end{align*}
\]
where the Green’s function(s) \( H^\alpha_{0+} (t, s), H^\beta_{0+} (s, \theta) ) \) are defined in [35].

**Definition 3.6.** [35] The integral (3.33) and (3.34) is HUS if there exist positive constants \( D_{1^*}, D_{2^*} \) fulfilling the conditions below:

For every \( \lambda_1, \lambda_2 > 0, \)
\[
\begin{align*}
|u(t) - \int_0^1 G_{\alpha_1} (t, s) \Phi_q \left( \int_0^1 G_{\beta_1} (s, \theta) \psi_1 (\theta, v(\theta)) d\theta \right) ds| &\leq \lambda_1, \\
|v(t) - \int_0^1 G_{\alpha_2} (t, s) \Phi_q \left( \int_0^1 G_{\beta_2} (s, \theta) \psi_2 (\theta, u(\theta)) d\theta \right) ds| &\leq \lambda_2. \quad \text{(3.35)}
\end{align*}
\]

There exists a pair, say \( (u^*(t), v^*(t)) \), satisfying
\[
\begin{align*}
u^*(t) &= \int_0^1 G_{\alpha_1} (t, s) \Phi_q \left( \int_0^1 G_{\beta_1} (s, \theta) \psi_1 (\theta, v^*(\theta)) d\theta \right) ds, \\
\psi^*(t) &= \int_0^1 G_{\alpha_2} (t, s) \Phi_q \left( \int_0^1 G_{\beta_2} (s, \theta) \psi_2 (\theta, u^*(\theta)) d\theta \right) ds. \quad \text{(3.36)}
\end{align*}
\]

Such that
\[
\begin{align*}
|u(t) - u^*(t)| &\leq D_{1^*} \lambda_1, \\
|v(t) - v^*(t)| &\leq D_{2^*} \lambda_2. \quad \text{(3.37)}
\end{align*}
\]

Khan et al also studied HUS for a coupled system of FDEs with initial and boundary conditions [36].

1. **Hyers-Ulam stability**
   
Here we study Hyers-Ulam stability of nonlinear system of FDEs with \( p \)-Laplacian operator (1.1). By taking help from Definition 3.6 and the work [35], we propose the following definition.

**Definition 4.1.** If there exist positive constants \( D_{1^*}, D_{2^*} \), then coupled systems of integral equations given by (3.12), (3.13) are Hyers-Ulam stable, satisfying:

For every \( \lambda_1, \lambda_2 > 0, \)
\[
\begin{align*}
u(t) - \int_0^1 H^\alpha_{0+} (t, s) \Phi_q \left( \int_0^1 H^\beta_{0+} (s, \theta) \psi_1 (\theta, v(\theta)) d\theta \right) ds \\
\psi(t) - \int_0^1 H^\alpha_{0+} (t, s) \Phi_q \left( \int_0^1 H^\beta_{0+} (s, \theta) \psi_2 (\theta, u(\theta)) d\theta \right) ds. \quad \text{(4.1)}
\end{align*}
\]
There exist a pair, say \((u'(t), v'(t))\), satisfying
\[
\begin{align*}
\left\{ 
\begin{array}{l}
u''(t) = \int_0^1 H^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H^{\beta_1}(s, \varphi) \psi_1(\varphi, v'(\varphi))d\varphi \right)ds, \\
v''(t) = \int_0^1 H^{\alpha_2}(t, s) \phi_q \left( \int_0^1 H^{\beta_2}(s, \varphi) \psi_2(\varphi, u'(\varphi))d\varphi \right)ds,
\end{array}
\right.
\end{align*}
\]
(4.2)
Such that
\[
\begin{align*}
|u(t) - u'(t)| &\leq D_1^1 \lambda_1, \\
|v(t) - v'(t)| &\leq D_2^2 \lambda_2.
\end{align*}
\]

**Theorem 4.2.** With the assumptions \((Q_1), (Q_2)\), solution of couple system of FDEs \(\phi_0\) (1.1), is Hyers-Ulam stable.

*Proof.* From Theorem 3.5 and definition 4.1, let \((u(t), v(t))\) be a solution of the system (3.12), (3.13). Let \((u'(t), v'(t))\) be any other approximation satisfying (4.2). Then, we have
\[
|u(t) - u'(t)| = \left| \int_0^1 H^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H^{\beta_1}(s, \varphi) \psi_1(\varphi, v'(\varphi))d\varphi \right)ds - \int_0^1 H^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H^{\beta_1}(s, \varphi) \psi_1(\varphi, v''(\varphi))d\varphi \right)ds \right|
\]
\[
\leq |(p - 1)\sigma^{p-2}(1)\left| \int_0^1 H^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H^{\beta_1}(s, \varphi) \psi_1(\varphi, v''(\varphi))d\varphi \right)ds - \int_0^1 H^{\alpha_1}(t, s) \phi_q \left( \int_0^1 H^{\beta_1}(s, \varphi) \psi_1(\varphi, \varphi) \right)ds \right|
\]
\[
\leq |(p - 1)\sigma^{p-2}\lambda_1 \left( \frac{1}{\Gamma(\alpha_1 + 1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)} \right) (\frac{1}{\Gamma(\beta_1 + 1)} + \frac{1}{\Gamma(\beta_1 + 1)}) \| v(t) - v'(t) \|
\]
\[
= \vartheta_1 \| v(t) - v'(t) \|,
\]
(4.4)

And
\[
|v(t) - v'(t)| = \left| \int_0^1 H^{\alpha_2}(t, s) \phi_q \left( \int_0^1 H^{\beta_2}(s, \varphi) \psi_2(\varphi, u'(\varphi))d\varphi \right)ds - \int_0^1 H^{\alpha_2}(t, s) \phi_q \left( \int_0^1 H^{\beta_2}(s, \varphi) \psi_2(\varphi, u''(\varphi))d\varphi \right)ds \right|
\]
\[
\leq |(p - 1)\sigma^{p-2}(1)\left| \int_0^1 H^{\alpha_2}(t, s) \phi_q \left( \int_0^1 H^{\beta_2}(s, \varphi) \psi_2(\varphi, u''(\varphi))d\varphi \right)ds - \int_0^1 H^{\alpha_2}(t, s) \phi_q \left( \int_0^1 H^{\beta_2}(s, \varphi) \psi_2(\varphi, \varphi) \right)ds \right|
\]
\[
\leq |(p - 1)\sigma^{p-2}\lambda_2 \left( \frac{1}{\Gamma(\alpha_2 + 1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_2 - m + 2)} \right) (\frac{1}{\Gamma(\beta_2 + 1)} + \frac{1}{\Gamma(\beta_2 + 1)}) \| u(t) - u'(t) \|
\]
\[
= \vartheta_2 \| u(t) - u'(t) \|,
\]
(4.5)
Where \(D_1^1 = \vartheta_1, D_2^2 = \vartheta_2\). Hence, by the help of (4.4), (4.5) the system (3.12), (3.13), is Hyers-Ulam stable. Therefore, Eq. (1.1) is Hyers-Ulam stable.

**Conclusion**

We have considered a high order coupled system of FDEs with nonlinear p-Laplacian operator for the examination of existence, uniqueness of solution and Hyer-Ulam stability by using topological degree theory. For these aims, we transformed the supposed problem into an integral system via Green’s function(s) and assumed certain necessary conditions over a Banach space. Our results are more general and useful than the standard case.

**Author’s Contributions**

All the authors of the paper have equal contribution in the paper. They have read and approved the paper before submission for publication.

**References**


Kiran Tanabbass: She was born in Pakistan on August 3, 1993. She did her bachelor’s degree (2011-2015) in statistics with minor in mathematics and computer science from University of the Punjab, Lahore, Pakistan. She is currently studying her master’s degree in mathematics from College of Science, Hohai University, Nanjing, Jiangsu, P.R. China. Her research work interest involves in the existence and uniqueness of solution for complex and difficult fractional differential equations involving $p$-Laplacian operator. This work is nowadays a hot topic for researchers belonging to engineering and mathematics. In some of complex problems, the question raises whether the system is stable or not. To overcome this question I also intend to investigate Hyers-Ulam stability for more complex problems.