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Existence Criteria and Hyers-Ulam Theorem for a Coupled P-Laplacian System of Fractional Differential Equations

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***Abstract*—** Dealing with high order coupled systems of FDEs through nonlinear p-Laplacian operator. We analyze existence, uniqueness & Hyer-Ulam stability (HUS) of the solutions by means of topological degree method. For this purpose, we transform the supposed problem into an integral system via Green's function(s) and assume certain operator equivalent to the integral form of the problem. Then after, the results are proved with some necessary assumptions.

***Keywords*—** Fractional differential equations (FDEs), Hyer-Ulam stability (HUS), topological degree theory, existence and uniqueness of solutions (EUS).

I. Introduction

The real world physical phenomena described by mathematical models of fractional differential equations (FDEs) are more constructive and practical in memory as compared to the models of integer order differential equations. . Due to the application of FDEs, one can learn fractional calculus in diverse fields like metallurgy, signal and image processing, economics, fractal theory, biology and other disciplines [16-26]. Existence of solutions for FDEs is one of the most attracted research areas. For the different classes of FDEs, one can study different methods for existence and uniqueness of solutions. Various nonlinear mathematical models can be found in the scientific fields to study dynamic systems. The classical nonlinear operator ϕ_p is one of the most important and frequently used nonlinear operators, which satisfies

$$\frac{1}{p}+\frac{1}{q}=1,\quad \phi_p(s)=|s|^{p-2}s,\quad p>1 \text{ and} \phi_q(\theta)=\phi_p^{-1}(\theta).$$

For details and applications of nonlinear operator ϕ_p, we pass on readers to [27-34].

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Here we highlight some related and interested research problems and contribution of scientists. Baleanu et al. [1] proved existence of some super linear FDE solutions and presented some applications of their results. Kuman et al. [2] presented stability and existence results for a class of FDE with help of topological degree theory. Baleanu et al. [3] proved existence solution for a nonlinear FDE on partially ordered Banach spaces. Baleanu et al. [4] studied that under certain assumptions the solutions of FDEs are eventually large and eventually small. Mahmudov and unul [5] studied a FDE with integral conditions involving order $2<\alpha\leq 3$, an impulsive fractional differential equation [6] and FDE with p-Laplacian operator [7], for existence of solutions.

Hu et al. [8] calculated existence of non-linear FDEs using the p-Laplacian operator:

$$\begin{cases} D_{0+}^{\gamma}\left(\phi_p\left(D_{0+}^{\rho}\mu(t)\right)\right)+f\left(x,\mu(t),D_{0+}^{\rho}\mu(t)\right)=0, & t\in(0,1),\\ D_{0+}^{\rho}\mu(0)=0=D_{0+}^{\rho}\mu(1), \end{cases}$$

where 0<ρ,γ<1,1<ρ+γ<2,D_(0+)^ρ,D_(0+)^γ are in the sense of Caputo derivatives.

Ali *et al.* [9] calculated the EUS and HUS for coupled system of FDEs:

$$\begin{aligned} &D_{0+}^{\rho}x(t)=f\big(t,y(t)\big), && t\in[0,1],\\ &D_{0+}^{\gamma}y(t)=f\big(t,x(t)\big), && t\in[0,1],\\ &x(0)=0,\quad x(t)|_{t=1}=\frac{1}{\Gamma(\sigma)}\int_0^T(T-s)^{\sigma-1}p(x(s))\,ds,\\ &y(0)=0,\quad y(t)|_{t=1}=\frac{1}{\Gamma(\delta)}\int_0^T(T-s)^{\delta-1}q(y(s))\,ds, \end{aligned}$$

where $\rho,\gamma,\sigma,\delta\in(1,2], D_{0+}^{\rho}, D_{0+}^{\gamma}$ are in the sense of Caputo derivatives, $p,q\in L[0,1]$.

Khan *et al.* [36] recently calculated the existence and uniqueness of positive solutions and HUS for the following system of coupled FDEs:

$$D_{0+}^{\gamma_1}\left(\phi_p\left(D_{0+}^{\rho_1}x(t)\right)\right)=-\Psi_1\big(t,y(t)\big),\ D_{0+}^{\gamma_2}\left(\phi_p\left(D_{0+}^{\rho_2}y(t)\right)\right)=-\Psi_2\big(t,x(t)\big),$$

$$D_{0+}^{\rho_1}x(0)=0=\left(\phi_p\left(D_{0+}^{\rho_1}x(t)\right)\right)'|_{t=0}=D_{0+}^{\delta_1}x(t)|_{t=\eta_1}, x(1)=\frac{\Gamma(2-\delta_1)}{{\eta_1}^{1-\delta_1}}\mathcal{J}^{\rho_1-\delta_1}\phi_q\left(\mathcal{J}_{0+}^{\gamma_1}\Psi_1\big(t,y(t)\big)\right)|_{t=\eta_1},$$

$$D_{0+}^{\rho_2}y(0)=0=\left(\phi_p\left(D_{0+}^{\rho_2}y(t)\right)\right)'|_{t=0}=D_{0+}^{\delta_2}y(t)|_{t=\eta_2}, y(1)=\frac{\Gamma(2-\delta_2)}{{\eta_2}^{1-\delta_2}}\mathcal{J}^{\rho_2-\delta_2}\phi_q\left(\mathcal{J}_{0+}^{\gamma_2}\Psi_2(t,x(t))\right)|_{t=\eta_2}$$

where $t\in[0,1], \rho_i,\gamma_i\in(1,2], \eta_i,\delta_i\in(0,1]$, for $i=1,2$, $and\ D_{0+}^{\rho_i}, D_{0+}^{\gamma_i}, D_{0+}^{\delta_i}$ for $i=1,2$ are denoting Caputo fractional derivatives.

In classical cases the fixed point theorems have some very strong conditions which limit the applications of the results to big extent for the study of many categories of FDEs and their coupled systems. Nowadays, degree theory plays a significant role in relaxing of the necessary conditions required for the study of fixed points of operators and EUS for a large number of FDEs and its coupled systems for their solutions. Different types of degree theorems have been produced including the well-known Brouwer and Leray-Schauder theory which have been considered by a large number of scientists for the exploration of different aspects of fractional calculus especially dealing with existence of positive solution of differential equations involving integer order as well non-integer. A version of the degree theory acknowledged as topological degree theory which was introduced by Mawhin [10] and further expanded by Isaia [11] was considered for the existence results of linear as well nonlinear FDEs. The proposed technique is also known as prior estimation technique, which does not involve the compactness of operators. For new results on topological degree theory, we suggest the readers for the study of some recently developed results in [12-14].

Enthused from the abovementioned studies, we study the EUS and HUS of a coupled system with initial and boundary conditions and non-linear operator ϕ_p using the topological degree method:

$$\begin{cases} D_{0+}^{\beta_1}\left(\phi_p\left(D_{0+}^{\alpha_1}u(t)\right)\right)=-\psi_1(t,v(t)),\ D_{0+}^{\beta_2}\left(\phi_p\left(D_{0+}^{\alpha_2}v(t)\right)\right)=-\psi_2(t,u(t)),\\ \left(\phi_p\left(D_{0+}^{\alpha_1}u(t)\right)\right)|_{t=1}=0, \left(\phi_p\left(D_{0+}^{\alpha_1}u(t)\right)\right)^{(k)}|_{t=0}=0, \text{for } k=1,2,3,\dots,n-1,\\ \left(\phi_p\left(D_{0+}^{\alpha_2}v(t)\right)\right)|_{t=1}=0, \left(\phi_p\left(D_{0+}^{\alpha_2}v(t)\right)\right)^{(k)}|_{t=0}=0, \text{for } k=1,2,3,\dots,n-1,\\ u^{(i)}(0)=0, \text{for } i=0,1,2,\dots.,m-2,m,\dots.,n-1, u^{(m-1)}(1)=0,\\ v^{(i)}(0)=0, \text{for } i=0,1,2,\dots.,m-2,m,\dots.,n-1, v^{(m-1)}(1)=0, \end{cases} \tag{1.1}$$

where $\alpha_i,\beta_i\in(n-1,n], \psi_1,\psi_2\in L[0,1]$ and $D_{0+}^{\alpha_i}, D_{0+}^{\beta_i}$ for $i=1,2$ stand for Caputo fractional derivative $\phi_p(\mathcal{N})=|\mathcal{N}|^{p-2}\mathcal{N}$ is non-linear operator ϕ_p satisfying $1/p+1/q=1, \phi_q$ represents inverse of ϕ_p. Here, we affirm that applying degree method to treat existence, uniqueness also to get the conditions for the stability of Hyers-ULam to a coupled system of (FDEs) with ϕ_p (1.1) has not been explored to our loyal awareness and understanding. Therefore, this work may get the attention of researchers to the study of Hyers-Ulam stability for more complex problems. We test enough conditions for the stability of EUS and HUS for system (1.1).

II. Auxiliary Results

Definition 2.1. The fractional integral of order $\alpha>0$ of $f:(0,+\infty)\to\mathbb{R}$ is defined as

$$I_{0+}^{\alpha}\psi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\psi(s)ds,$$

Given that integral on R.H.S is point wise defined on the interval $(0,+\infty)$, where

$$\Gamma(\alpha)=\int_0^{+\infty}e^{-s}s^{\alpha-1}ds.$$

Definition 2.2. For a function $\psi(t)$ the fractional Caputo's derivative of α order is defined by the following integral form (provided it exist)

$$D_{0+}^{\alpha}\psi(t)=\frac{1}{\Gamma([\alpha]+1-\alpha)}\int_0^t(t-s)^{[\alpha]-\alpha}\,\psi^{([\alpha]+1)}(s)ds,$$

Where $[\alpha]$ is the integer part of α.

Lemma 2.3. Let $\alpha\in(n-1,n], \psi\in AC^{n-1}$, then

$$I_{0_+}^{\alpha}D_{0+}^{\alpha}\psi(t)=\psi(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

For the $c_i\in\mathbb{R}$ for $i=0,1,2,\dots,n-1$.

Consider the space of real valued continuous functions $V=C([0,1],\mathbb{R})$ with Topological norm $\|v\|=\sup\{|v(t)|:0\le t\le 1$ for $v\in V$. The product space $\omega*=V\times V$ with the norm $\|(v,z)\|=\|v\|+\|z\|$ is also Banach space. We give a notation S to the class of all bounded mappings in ω.

Definition 2.4. For the mapping $\xi:S\to(0,\infty)$ Kuratowski measure of non-compactness is:

$$\xi(\mathcal{M})=\inf\{r>0:\mathcal{M}\text{ the finite cover for sets of diameter }\le r\}$$

where $\mathcal{M}\in S$.

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Definition 2.5. Let $T: \vartheta \to V$ be bounded and continuous mapping with $\vartheta \subset V$. Then T is an ξ −Lipschitz, where $\varsigma \geq 0$ if

$$\xi\big(T(\mathcal{M})\big) \leq \varsigma\xi(\mathcal{M}) \text{for all bounded } \mathcal{M} \subset \vartheta.$$

And T is a strict ξ-contraction with $\varsigma < 1$.

Definition 2.6. The function T is ξ-condensing if

$$\xi\big(T(\mathcal{M})\big) < \xi(\mathcal{M}) \text{for all bounded } \mathcal{M} \subset \vartheta \text{ such that } \xi(\mathcal{M}) > 0.$$

Therefore $\xi\big(T(\mathcal{M})\big) \geq \xi(\mathcal{M})$yields $\xi(\mathcal{M}) = 0$.

More we have $T: \vartheta \to V$ is Lipschitz for $\varsigma > 0$, such that

$$\| T(v) - T(\bar{v}) \| \leq \varsigma \| v - \bar{v} \| \quad \text{for all } v, \bar{v} \in \vartheta.$$

The condition $\varsigma < 1$, then T is a strict contraction.

Proposition 2.7. The T is said to be ξ-Lipschitz with $\varsigma = 0$ if and only if $T: \vartheta \to V$ is said to be compact.

Proposition 2.8. The T is said to be ξ-Lipschitz for constant value ς if and only if $T: \vartheta \to V$, is Lipschitz with constant ς.

Theorem 2.9. [11] Let $T: V \to V$ be an ξ-condensing and

$$\mathcal{H} = \{z \in V: \text{there exist } 0 \leq \lambda \leq 1 \text{ such that } z = \lambda Tz\}.$$

If $\mathcal{H}$ is a bounded in V, there exists $a > 0$, and $\mathcal{H} \subset \mathcal{M}_a(0)$,with degree

$$deg(I - \lambda\mathcal{G}, \mathcal{M}_r(0), 0) = 1 \text{ for every } \lambda \in [0,1].$$

So, T has atleast one fixed point and collection of all fixed points of T are contained in $\mathcal{M}_a(0)$.

Lemma 2.10 [15] For the nonlinear operator ϕ_p , we have

1. If $1 < p \leq 2, \ell_1\ell_2 > 0$ and$|\ell_1|, |\ell_2| \geq m > 0$, then $|\phi_p(\ell_1) - \phi_p(\ell_2)| \leq (p-1)m^{p-2}|\ell_1 - \ell_2|$.
2. If $p > 2$, and $|\ell_1|, |\ell_1| \leq F$, then $|\phi_p(\ell_1) - \phi_p(\ell_2)| \leq (p-1)F^{p-2}|\ell_1 - \ell_2|$

III. MAIN RESULTS

Theorem 3.1. Let $\psi_1 \in C[0,1]$ be an integrable function satisfying (1.1). Then the solution of

$$\begin{cases} D_{0+}^{\beta_1}\left(\phi_p\left(D_{0+}^{\alpha_1}u(t)\right)\right) = -\psi_1(t, v(t)), \\ \left(\phi_p\left(D_{0+}^{\alpha_1}u(t)\right)\right)|_{t=1} = 0, \left(\phi_p\left(D_{0+}^{\alpha_1}u(t)\right)\right)^{(k)}|_{t=0} = 0, \text{for } k = 1,2,3,\dots,n-1 \\ u^{(i)}(0) = 0, \text{for} i = 0,1,2,\dots,m-2, m,\dots,n-1, u^{(m-1)}(1) = 0, \end{cases}$$

is given by the integral equation

$$u(x) = \int_0^1 H^{\alpha_1}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,t)\psi_1(t, v(t))dt\right)ds, \tag{3.2}$$

where $H^{\alpha_1}(t,s), H^{\beta_1}(t,s)$ are Green's function(s) defined by

$$H^{\alpha_1}(t,s) = \begin{cases} \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} - t^{m-1}\frac{(1-s)^{\alpha_1-m}}{\Gamma(m)\Gamma(\alpha_1-(m-1))}, & 0 \leq s \leq t \leq 1, \\ -t^{m-1}\frac{(1-s)^{\alpha_1-m}}{\Gamma(m)\Gamma(\alpha_1-(m-1))}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.3}$$

$$H^{\beta_1}(t,s) = \begin{cases} -\frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)} + \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.4}$$

Proof. Applying operator $I_{0+}^{\beta_1}$ on (3.1) and with help of Lemma (2.3), we proceed the following

$$\phi_p\left(D_{0+}^{\alpha_1}u(t)\right) = -I_{0+}^{\beta_1}\psi_1(t, v(t)) + c_1 + c_2t + \cdots + c_nt^{n-1} \tag{3.5}$$

Condition $\left(\phi_p\left(D_{0+}^{\alpha_1}u(t)\right)\right)^{(k)}|_{t=0} = 0, for\ k = 1,2,3,\dots,n-1$ results $c_2 = c_3 = \cdots = c_n = 0$. And $\left(\phi_p\left(D_{0+}^{\alpha_1}u(t)\right)\right)|_{t=1} = 0$, implies

$$c_1 = I_{0+}^{\beta_1}\psi_1(t, v(t))|_{t=1} = \frac{1}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}\psi_1(s, v(s))\,ds. \tag{3.6}$$

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By using values of $c_i\ for\ i = 1,2, \dots, n$ and (3.5), we have

$$\phi_p\left(D_{0+}^{\alpha_1}u(t)\right) = -I_{0+}^{\beta_1}\psi_1(t,v(t)) + I_{0+}^{\beta_1}\psi_1(t,v(t))|_{t=1}$$

$$= -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta_1-1}\,\psi_1(s,v(s))ds + \frac{1}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}\psi_1(s,v(s))\,ds$$

$$= \int_0^1 H^{\beta_1}(t,s)\,\psi_1(s,v(s))ds, \tag{3.7}$$

where$H^{\beta_1}(t,s)$ is a Green's function given in (3.4).

Applying $\phi_q = {\phi_p}^{-1}$in (3.7), we get

$$D_{0+}^{\alpha_1}u(t) = \phi_q\left(\int_0^1 H^{\beta_1}(t,s)\,\psi_1(s,v(s))ds\right). \tag{3.8}$$

Applying operator $I_{0+}^{\alpha_1}$ on (3.8) and Lemma 2.10, we get

$$u(t) = I_{0+}^{\alpha_1}\left(\phi_q\left(\int_0^1 H^{\beta_1}(t,s)\,\psi_1(s,v(s))ds\right)\right) + z_1 + z_2 t + \dots + z_m t^{m-1} + \dots + z_n t^{n-1} \tag{3.9}$$

Using condition $u^{(i)}(0) = 0, for\ i = 0,1,2, \dots, m-2, m, \dots, n-1$, we obtain $z_1 = z_2 = \cdots z_{m-1} = z_{m+1} = \cdots z_n = 0$. From condition $u^{(m-1)}(1) = 0$, we have

$$z_m = -\frac{1}{(m-1)!} I_{0+}^{\alpha_1-(m-1)}\left(\phi_q\left(\int_0^1 H^{\beta_1}(t,s)\,\psi_1(s,v(s))ds\right)\right)|_{t=1} \tag{3.10}$$

Now putting the values of $z_i, for\ i = 1,2, \dots, m, \dots, n$ in (3.9), we have

$$u(t) = I_{0+}^{\alpha_1}\left(\phi_q\left(\int_0^1 H^{\beta_1}(t,s)\,\psi_1(s,v(s))ds\right)\right) - \frac{t^{m-1}}{(m-1)!} I_{0+}^{\alpha_1-(m-1)}\left(\phi_q\left(\int_0^1 H^{\beta_1}(t,s)\,\psi_1(s,v(s))ds\right)\right)|_{t=1}$$

$$= \left(\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1} - \frac{t^{m-1}}{\Gamma(m)}\int_0^1 \frac{(1-s)^{\alpha_1-m}}{\Gamma(\alpha_1-(m-1))}\right)\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1(\wp,v(\wp))d\wp ds\right)$$

$$= \int_0^1 H^{\alpha_1}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1(\wp,v(\wp))d\wp\right)ds \tag{3.11}$$

where$H^{\alpha_1}(t,s), H^{\beta_1}(s,\wp)$ are defined by (3.3), (3.4), respectively, as Green's function(s).

Following Theorem 3.1, we may write our problem in the following system

$$u(t) = \int_0^1 H^{\alpha_1}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1(\wp,v(\wp))d\wp\right)ds, \tag{3.12}$$

$$v(t) = \int_0^1 H^{\alpha_2}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\,\psi_2(\wp,u(\wp))d\wp\right)ds, \tag{3.13}$$

where $H^{\alpha_2}(t,s), H^{\beta_2}(s,t)$ are Green's function(s) defined by

$$H^{\alpha_2}(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} - t^{m-1}\dfrac{(1-s)^{\alpha_2-m}}{\Gamma(m)\Gamma\big(\alpha_2-(m-1)\big)}, & 0 \le s \le t \le 1, \\ -t^{m-1}\dfrac{(1-s)^{\alpha_2-m}}{\Gamma(m)\Gamma\big(\alpha_2-(m-1)\big)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.14}$$

$$H^{\beta_2}(t,s) = \begin{cases} -\dfrac{(t-s)^{\beta_2-1}}{\Gamma(\beta_2)} + \dfrac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}, & 0 \le s \le t \le 1, \\ \dfrac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.15}$$

Define $T_i^*: V \to V for (i = 1,2)$ by

$$T_1^* u(t) = \int_0^1 H^{\alpha_1}(t,s)\, \phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1(\wp, v(\wp)) d\wp\right) ds \tag{3.16}$$

$$T_2^* v(t) = \int_0^1 H^{\alpha_2}(t,s)\, \phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2(\wp, u(\wp)) d\wp\right) ds \tag{3.17}$$

We further define $F(u,v) = (T_1^*(u), T_2^*(v))$. Then, with help of Theorem 3.1, the solution of (3.12), (3.13) is equivalent to any fixed point, say (x,y), of operator equation

$$(x,y) = F(x,y). \tag{3.18}$$

To proceed further, we need following assumptions in the main results of the paper.

(Q_1) With positive constant values of $a_1, b_1, \mathbb{M}_{\psi_1}^*, \mathbb{M}_{\psi_2}^* and\ k_1, k_2 \in [0,1]$, functions ψ_1, ψ_2 satisfy the following growth conditions

$$|\psi_1(x,v)| \le \phi_p(a_1|v|^{k_1} + \mathbb{M}_{\psi_1}^*),$$

$$|\psi_2(x,u)| \le \phi_p(b_1|u|^{k_2} + \mathbb{M}_{\psi_2}^*).$$

(Q_2) There exist real valued constants $\lambda_{\psi_1}, \lambda_{\psi_2}$ such that for all $u, v, x, y \in V$,

$$|\psi_1(t,v) - \psi_1(t,x)| \le \lambda_{\psi_1}|v - x|,$$

$$|\psi_2(t,u) - \psi_2(t,y)| \le \lambda_{\psi_2}|u - y|.$$

For simplicity in calculations, we define the following terms:

$$\Upsilon_1 = \left(\frac{1}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)}\right)\left(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(\beta_1+1)}\right)^{q-1},$$

$$\Upsilon_2 = \left(\frac{1}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_2 - m + 2)}\right)\left(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(\beta_2+1)}\right)^{q-1},$$

$$\rho_1^* = (\Upsilon_1 + \Upsilon_2)(\mathbb{M}_{\psi_1}^* + \mathbb{M}_{\psi_2}^*), \qquad \delta^* = (a_1 + b_1)(\Upsilon_1 + \Upsilon_2),$$

$$\vartheta_1 = (p-1)\sigma^{p-2}\lambda_{\psi_1}\left(\frac{1}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)}\right)\left(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(\beta_1+1)}\right),$$

$$\vartheta_2 = (p-1)\sigma^{p-2}\lambda_{\psi_2}\left(\frac{1}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_2 - m + 2)}\right)\left(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(\beta_2+1)}\right).$$

Theorem 3.2. With assumption (Q_1), the operator $F: \omega^* \to \omega^*$ is continuous and satisfies following growth condition

$$F(u,v)(t) \le \delta^* \parallel (u,v) \parallel^k + \rho_1^* \tag{3.19}$$

For each $(u,v) \in \mathcal{M}_r \subset \omega^*$

Proof. Consider a bounded set $\mathcal{M}_r = \{(u,v) \in \omega: \parallel (u,v) \parallel \le r\}$ with sequence $\{(u_n, v_n)\}$ converging to (u,v) in $\mathcal{M}_r$. To show that $\parallel T^*(u_n, v_n) - T^*(u,v) \parallel \to 0\ as\ n \to \infty$. Let us consider

$$\begin{aligned} |T_1^* u_n(t) - T_1^* u(t)| &= \left|\int_0^1 H^{\alpha_1}(t,s)\, \phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1(\wp, v_n(\wp)) d\wp\right) ds - \int_0^1 H^{\alpha_1}(t,s)\, \phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1(\wp, v(\wp)) d\wp\right) ds\right| \\ &\le \int_0^1 |H^{\alpha_1}(t,s)| \left|\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1(\wp, v_n(\wp)) d\wp\right) ds \right. \\ &\left. - \phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1(\wp, v(\wp)) d\wp\right)\right| ds, \end{aligned} \tag{3.20}$$

And

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$$
\begin{aligned}
&|T_2^* v_n(t) - T_2^* v(t)| \\
&\qquad = \left| \int_0^1 H^{\alpha_2}(t,s)\, \phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2\big(\wp, u_n(\wp)\big) d\wp\right) ds \right. \\
&\qquad \left. - \int_0^1 H^{\alpha_2}(t,s)\, \phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2\big(\wp, u(\wp)\big) d\wp\right) ds \right| \\
&\qquad \le \int_0^1 |H^{\alpha_2}(t,s)| \left| \phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2\big(\wp, u_n(\wp)\big) d\wp\right) \right. \\
&\qquad \left. - \phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2\big(\wp, u(\wp)\big) d\wp\right) \right| ds. \qquad (3.21)
\end{aligned}
$$

With the help of (3.20), (3.21) and due to the continuity of the functions ψ_1, ψ_2 we have $|T_1^* u_n(t) - T_1^* u(t)| \to 0 \ as \ n \to \infty$.This implies that T_1^* is continuous. Similarly, $|T_2^* v_n(t) - T_2^* v(t)| \to 0 \ as \ n \to \infty$, which implies T_2^* is continuous. The continuity of $T_1^* \ and \ T_2^*$ implies that operator $F = (T_1^*, T_2^*)$ is continuous.

Now, for the inequality (3.19), from (3.16), (3.17) and assumption (Q_1), we continue as

$$
\begin{aligned}
|T_1^* u(t)| &= \left| \int_0^1 H^{\alpha_1}(t,s)\, \phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1\big(\wp, v(\wp)\big) d\wp\right) ds \right| \\
&\le \int_0^1 |H^{\alpha_1}(t,s)|\, \phi_q\left(\int_0^1 |H^{\beta_1}(s,\wp)|\, |\psi_1\big(\wp, v(\wp)\big)| d\wp\right) ds \qquad (3.22) \\
&\le \int_0^1 |H^{\alpha_1}(t,s)|\, \phi_q\left(\int_0^1 |H^{\beta_1}(s,\wp)| \phi_p(a_1 \parallel v \parallel^{k_1} + \mathbb{M}_{\psi_1}^*) d\wp\right) ds \\
&\le \left(\frac{1}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)}\right)\left(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(\beta_1+1)}\right)^{q-1} \left(a_1 \parallel v \parallel^{k_1} + \mathbb{M}_{\psi_1}^*\right) \\
&= \Upsilon_1\big(a_1 \parallel v \parallel^{k_1} + \mathbb{M}_{\psi_1}^*\big).
\end{aligned}
$$

And

$$
\begin{aligned}
|T_2^* v(t)| &= \left| \int_0^1 H^{\alpha_2}(t,s)\, \phi_q\left(\int_0^1 |H^{\beta_2}(s,\wp)|\, |\psi_2\big(\wp, u(\wp)\big)| d\wp\right) ds \right| \\
&\le \int_0^1 |H^{\alpha_2}(t,s)|\, \phi_q\left(\int_0^1 |H^{\beta_2}(s,\wp)|\, |\psi_2\big(\wp, u(\wp)\big)| d\wp\right) ds \qquad (3.23) \\
&\le \int_0^1 |H^{\alpha_2}(t,s)|\, \phi_q\left(\int_0^1 |H^{\beta_2}(s,\wp)|\, \phi_p(b_1 |u|^{k_2} + \mathbb{M}_{\psi_2}^*) d\wp\right) ds \\
&\le \left(\frac{1}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_2 - m + 2)}\right)\left(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(\beta_2+1)}\right)^{q-1} (b_1 \parallel u \parallel^{k_2} + \mathbb{M}_{\psi_2}^*) \\
&= \Upsilon_2(b_1 |u|^{k_2} + \mathbb{M}_{\psi_2}^*)
\end{aligned}
$$

With the help of (3.22), (3.23), we proceed

$$
\begin{aligned}
|F^*(u,v)(t)| &\le \Upsilon_1\big(a_1 \parallel v \parallel^{k_1} + \mathbb{M}_{\psi_1}^*\big) + \Upsilon_2(b_1 \parallel u \parallel^{k_2} + \mathbb{M}_{\psi_2}^*) \\
&\le \delta^* \parallel (u,v) \parallel^{k} + \rho_1^* \qquad (3.24)
\end{aligned}
$$

This completes the proof.

Theorem 3.3. Assumethat (Q_1) holds true. Then$F^*: \omega^* \to \omega^*$ is compact and $\xi -$ Lipschitzwith constant zero.

Proof. Theorem 3.2 implies that$F^*: \omega \to \omega$ is bounded. Next, let $\mathcal{B} \subset \mathcal{M}_r \subset \omega^*$. Then, by (Q_1), Lemma 3.1, Eq. (3.12), (3.13), then for any $t_1, t_2 \in [0,1]$, we have

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$$|T_1^*u(t_1) - T_1^*u(t_2)|$$
$$= \left| \int_0^1 H^{\alpha_1}(t_1,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1\big(\wp, v(\wp)\big)d\wp\right)ds - \int_0^1 H^{\alpha_1}(t_2,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1\big(\wp, v(\wp)\big)d\wp\right)ds \right| \tag{3.25}$$
$$\leq \int_0^1 |H^{\alpha_1}(t_1,s) - g^{\alpha_1}(t_2,s)|\phi_q\left(\int_0^1 |H^{\beta_1}(s,\wp)|\,\phi_p(a_1 \parallel v \parallel^{k_1} + \mathbb{M}_{\psi_1}^*)d\wp\right)ds$$
$$\leq \left(\frac{|{t_1}^{\alpha_1} - {t_2}^{\alpha_1}|}{\Gamma(\alpha_1+1)} + \frac{|{t_1}^{m-1} - {t_2}^{m-1}|}{\Gamma(m)\Gamma(\alpha_1-m+2)}\right)\left(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(\beta_1+1)}\right)^{q-1}\big(a_1 \parallel v \parallel^{k_1} + \mathbb{M}_{\psi_1}^*\big),$$

$$|T_2^*v(t_1) - T_2^*v(t_2)|$$
$$= \left| \int_0^1 H^{\alpha_2}(t_1,s)\,\phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\,\psi_2\big(\wp, u(\wp)\big)d\wp\right)ds - \int_0^1 H^{\alpha_2}(t_1,s)\,\phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\,\psi_2\big(\wp, u(\wp)\big)d\wp\right)ds \right| \tag{3.26}$$
$$\leq \int_0^1 |H^{\alpha_2}(t_1,s) - H^{\alpha_2}(t_1,s)|\phi_q\left(\int_0^1 |H^{\beta_2}(s,\wp)|\,\phi_p(b_1|u|^{k_2} + \mathbb{M}_{\psi_2}^*)d\wp\right)ds$$
$$\leq \left(\frac{|{t_1}^{\alpha_2} - {t_2}^{\alpha_2}|}{\Gamma(\alpha_2+1)} + \frac{|{t_1}^{m-1} - {t_2}^{m-1}|}{\Gamma(m)\Gamma(\alpha_2-m+2)}\right)\left(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(\beta_2+1)}\right)^{q-1} b_1|u|^{k_2} + \mathbb{M}_{\psi_2}^*),$$

From (3.25), (3.26), we have

$$|F^*(u,v)(t_1) - F^*(u,v)(t_2)|$$
$$\leq \left(\frac{|{t_1}^{\alpha_1} - {t_2}^{\alpha_1}|}{\Gamma(\alpha_1+1)} + \frac{|{t_1}^{m-1} - {t_2}^{m-1}|}{\Gamma(m)\Gamma(\alpha_1-m+2)}\right)\left(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(\beta_1+1)}\right)^{q-1}\big(a_1 \parallel v \parallel^{k_1} + \mathbb{M}_{\psi_1}^*\big)$$
$$+ \left(\frac{|{t_1}^{\alpha_2} - {t_2}^{\alpha_2}|}{\Gamma(\alpha_2+1)} + \frac{|{t_1}^{m-1} - {t_2}^{m-1}|}{\Gamma(m)\Gamma(\alpha_2-m+2)}\right)\left(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(\beta_2+1)}\right)^{q-1}(b_1|u|^{k_2} + \mathbb{M}_{\psi_2}^* \tag{3.27}$$

As $t_1 \to t_2$, (3.27) approaches to zero which may be observe on the right side. Thus, the operator$F^* = (T_1^*, T_2^*)$ is an equi-continuous on $\mathcal{B}$. Arzela-Ascoli theorem implies that $F^*(\mathcal{B})$ is compact. Ultimately, $\mathcal{B}$ is $\xi -$ Lipschitz with constant zero.

Theorem 3.4. With assumtions $Q_1 - Q_2$and$\delta^* < 1$, the system of FDEs with ϕ_p (1.1) has a bounded solution in ω^*.

Proof. For existence of solution of the coupled differential system of fractional order (1.1), we take help from Theorem 2.9. Let us consider the set

$S = \{(u,v) \in \omega^* \colon \text{there exist } \lambda \in [0,1], \text{such that} (u,v) = \lambda F(u,v)\},$

We show that S is bounded. For this we assume a contrary path. Let for some $(u,v) \in S$, such that $\parallel (u,v) \parallel = \mathcal{J} \to \infty$. But from Theorem 3.2, we have

$$\parallel (u,v) \parallel = \parallel \lambda F(u,v) \parallel \leq \parallel F(u,v) \parallel \leq \delta^* \parallel (u,v) \parallel^k + \rho_1^* \tag{3.28}$$

Since $\parallel (u,v) \parallel = \mathcal{J}$, then (3.28) implies

$$\parallel (u,v) \parallel \leq \delta^* \parallel (u,v) \parallel^k + \rho_1^*$$
$$1 \leq \delta^* \frac{\parallel (u,v) \parallel^k}{\parallel (u,v) \parallel} + \frac{\rho_1^*}{\parallel (u,v) \parallel}$$
$$1 \leq \delta^* \frac{1}{\mathcal{J}^{1-k}} + \frac{\rho_1^*}{\mathcal{J}} \to 0, as\ \mathcal{J} \to \infty \qquad .$$

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This is a contradiction. Ultimately, $\| (u,v) \| < \infty$ and hence the set S is bounded and therefore, by Theorem 2.9, the operator F has atleast one fixed point which is the solution of the coupled system of FDEs (1.1). And the set of solution of (1.1) is bounded inω^*.

Theorem 3.5. Assume that $Q_1 - Q_2$ hold. Then Eq. (1.1) has a unique solution provided that $\vartheta_1 + \vartheta_2 < 1$.

Proof. From (3.16), (3.17) and assumptions $Q_1 - Q_2$ and Lemma (2.10). Then for any $t_1, t_2 \in [0,1]$, we proceed

$$|T_1^* u(t) - T_1^* \bar{u}(t)|$$
$$= \left| \int_0^1 H^{\alpha_1}(t,s)\, \phi_q \left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1\big(\wp, v(\wp)\big) d\wp \right) ds - \int_0^1 H^{\alpha_1}(t,s)\, \phi_q \left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1\big(\wp, \bar{v}(\wp)\big) d\wp \right) ds \right|$$
$$= \int_0^1 |H^{\alpha_1}(t,s)| \left| \phi_q \left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1\big(\wp, v(\wp)\big) d\wp ds \right) \right.$$
$$\left. - \phi_q \left(\int_0^1 H^{\beta_1}(s,\wp)\, \psi_1\big(\wp, \bar{v}(\wp)\big) d\wp \right) ds \right| \tag{3.29}$$
$$= (p-1)\sigma^{p-2} \int_0^1 |H^{\alpha_1}(t,s)| \int_0^1 |H^{\beta_1}(s,\wp)|\, \big|\psi_1\big(\wp, v(\wp)\big) - \psi_1\big(\wp, \bar{v}(\wp)\big)\big| d\wp ds$$
$$\leq (p-1)\sigma^{p-2} \lambda_{\psi_1} \left(\frac{1}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)} \right) \left(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(\beta_1+1)} \right) (|v(t) - \bar{v}(t)|),$$
$$= \vartheta_1(|v(t) - \bar{v}(t)|),$$

And

$$|T_2^* v(t) - T_2^* \bar{v}(t)|$$
$$= \left| \int_0^1 H^{\alpha_2}(t,s)\, \phi_q \left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2\big(\wp, u(\wp)\big) d\wp \right) ds - \int_0^1 H^{\alpha_2}(t,s)\, \phi_q \left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2\big(\wp, \bar{u}(\wp)\big) d\wp \right) ds \right|$$
$$= \int_0^1 |H^{\alpha_2}(t,s)| \left| \phi_q \left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2\big(\wp, u(\wp)\big) \right) d\wp ds \right.$$
$$\left. - \phi_q \left(\int_0^1 H^{\beta_2}(s,\wp)\, \psi_2\big(\wp, \bar{u}(\wp)\big) d\wp \right) ds \right| \tag{3.30}$$
$$= (p-1)\sigma^{p-2} \int_0^1 |H^{\alpha_2}(t,s) \left| \int_0^1 |H^{\beta_2}(s,\wp) \right| |\psi_2\big(\wp, u(\wp)\big) - \psi_2\big(\wp, \bar{u}(\wp)\big)| d\wp ds$$
$$\leq (p-1)\sigma^{p-2} \lambda_{\psi_2} \left(\frac{1}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_2 - m + 2)} \right) \left(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(\beta_2+1)} \right) \big(|u(t) - \bar{u}(t)\big)$$
$$= \vartheta_2\big(|u(t) - \bar{u}(t)\big).$$

From (3.29), (3.30), we have

$$|F^*(u,v)(t) - F^*(\bar{u},\bar{v})(t)| \leq \vartheta_1(|v(t) - \bar{v}(t)|) + \vartheta_2\big(|u(t) - \bar{u}(t)\big)$$
$$\leq (\vartheta_1 + \vartheta_2)(\| (u,v)(t) - (\bar{u},\bar{v})(t) \|). \tag{3.31}$$

With assumption $\vartheta_1 + \vartheta_2 < 1$, Banach's contraction principle implies that F^* has a unique fixed point. As a result, the solution of the system of fractional order with ϕ_p (1.1) is unique.

Recently, Khan *et al.*[35] studied the stability of Hyers-Ulam for the following system of FDEs with ϕ_p:

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$$
\begin{cases}
D_{0_+}^{\beta_1}\left(\phi_p\left(D_{0_+}^{\alpha_1}u(t)\right)\right)+\psi_1\big(t,v(t)\big)=0, \quad D_{0_+}^{\beta_2}\left(\phi_p\left(D_{0_+}^{\alpha_2}v(t)\right)\right)+\psi_2\big(t,u(t)\big)=0,\\
\left(\phi_p\left(D_{0_+}^{\alpha_1}u(t)\right)\right)|_{t=1}=I_{0_+}^{\beta_1-1}\left(\psi_1\big(t,v(t)\big)\right)|_{t=1},\\
\left(\phi_p\left(D_{0_+}^{\alpha_1}u(t)\right)\right)^{'}|_{t=1}=0=\left(\phi_p\left(D_{0_+}^{\alpha_1}u(t)\right)\right)^{''}|_{t=0,}\\
\left(\phi_p\left(D_{0_+}^{\alpha_2}v(t)\right)\right)|_{t=1}=I_{0_+}^{\beta_2-1}\left(\psi_2\big(t,u(t)\big)\right)|_{t=1},\\
\left(\phi_p\big(D_{0_+}^{\alpha_2}v(t)\big)\right)^{'}|_{t=1}=0=\left(\phi_p\big(D_{0_+}^{\alpha_2}v(t)\big)\right)^{''}|_{t=0},\\
u(0)=0=u^{''}(0), \qquad u(1)=0, \qquad v(0)=0=v^{''}(0), \quad v(1)=0,
\end{cases}
\tag{3.32}
$$

Where $2<\alpha_i,\beta_i<3, \psi_1,\psi_2 \in L[0,1], and\ D_{0_+}^{\alpha_i}, D_{0_+}^{\beta_i}\ for\ i=1,2$ are in Caputo sense.

They converted (3.32) system into following coupled system of Hammerstein-type integral equations given below:

$$u(t)=\int_0^1 \mathcal{H}^{\alpha_1}(t,s)\,\phi_q\left(\int_0^1 \mathcal{H}^{\beta_1}(s,\theta)\,\psi_1\big(\theta,v(\theta)\big)d\theta\right)ds, \tag{3.33}$$

$$v(t)=\int_0^1 \mathcal{H}^{\alpha_2}(t,s)\,\phi_q\left(\int_0^1 \mathcal{H}^{\beta_2}(s,\theta)\,\psi_2\big(\theta,u(\theta)\big)d\theta\right)ds, \tag{3.34}$$

where the Green's function(s) $\mathcal{H}^{\alpha_1}(t,s),\mathcal{H}^{\beta_1}(s,\theta),\mathcal{H}^{\alpha_2}(t,s),\mathcal{H}^{\beta_2}(s,\theta)$are defined in [35].

Definition 3.6. [35] The integral (3.33) and (3.34) is HUS if there exist positive constants $D_1^*,\ D_2^*$ fulfilling the conditions below: For every $\lambda_1,\lambda_2>0,$ if

$$|u(t)-\int_0^1 \mathcal{G}_{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}_{\beta_1}\psi_1\big(\tau,v(\tau)\big)\right)ds|\le\lambda_1,$$

$$|v(t)-\int_0^1 \mathcal{G}_{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}_{\beta_2}\psi_2\big(\tau,u(\tau)\big)\right)ds|\le\lambda_2, \tag{3.35}$$

There exists a pair, say $(u^*(t),v^*(t))$, satisfying

$$u^*(t)=\int_0^1 \mathcal{G}_{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}_{\beta_1}\,(s,\theta)\psi_1\big(\theta,v^*(\theta)\big)d\theta\right)ds,$$

$$v^*(t)=\int_0^1 \mathcal{G}_{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}_{\beta_2}\,(s,\theta)\psi_2\big(\theta,u^*(\theta)\big)d\theta\right)ds, \tag{3.36}$$

Such that

$$|u(t)-u^*(t)|\le D_1^*\lambda_1,$$

$$|v(t)-v^*(t)|\le D_2^*\lambda_2. \tag{3.37}$$

Khan.*et al* also studied HUS for a coupled system of FDEs with initial and boundary conditions [36].

1. Hyers-Ulam stability

Here we study Hyers-Ulam stability of nonlinear system of FDEs with *p*-Laplacian operator (1.1). By taking help from Definition 3.6 and the work [35], we propose the following definition.

Definition 4.1. If there exist positive constants D_1^*,D_2^* , then coupled systems of integral equations given by (3.12), (3.13) are Hyers-Ulam stable, satisfying:

For every $\lambda_1,\lambda_2>0$, if

$$
\begin{cases}
u(t)-\int_0^1 H^{\alpha_1}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1(\wp,v(\wp))d\wp\right)ds\\
v(t)-\int_0^1 H^{\alpha_2}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\,\psi_2(\wp,u(\wp))d\wp\right)ds,
\end{cases}
\tag{4.1}
$$

There exist a pair, say $(u^*(t), v^*(t))$, satisfying

$$\begin{cases} u^*(t) = \int_0^1 H^{\alpha_1}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1(\wp, v^*(\wp))d\wp\right)ds, \\ v^*(t) = \int_0^1 H^{\alpha_2}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\,\psi_2(\wp, u^*(\wp))d\wp\right)ds, \end{cases} \tag{4.2}$$

Such that

$$|u(t) - u^*(t)| \le D_1^*\lambda_1, \tag{4.3}$$
$$|v(t) - v^*(t)| \le D_2^*\lambda_2.$$

Theorem 4.2. With the assumptions (Q_1),(Q_2), solution of couple system of FDEs ϕ_p (1.1), is Hyers-Ulam stable.

Proof. From Theorem 3.5 and definition 4.1, let $(u(t), v(t))$ be a solution of the system (3.12), (3.13). Let $(u^*(t), v^*(t))$ be any other approximation satisfying (4.2). Then, we have

$$|u(t) - u^*(t)| = \left|\int_0^1 H^{\alpha_1}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1\big(\wp, v(\wp)\big)d\wp\right)ds - \int_0^1 H^{\alpha_1}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_1}(s,\wp)\,\psi_1(\wp, v^*(\wp))d\wp\right)\right| ds$$
$$\le (p-1)\sigma^{p-2}\left(\int_0^1 |H^{\alpha_1}(t,s)| \int_0^1 \left|H^{\beta_1}(s,\wp)\right| \left|\psi_1\big(\wp, v(\wp)\big) - \psi_1\big(\wp, v^*(\wp)\big)\right| d\wp ds\right) \tag{4.4}$$
$$\le (p-1)\sigma^{p-2}\lambda_{\psi_1}\left(\frac{1}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_1 - m + 2)}\right)\left(\frac{1}{\Gamma(\beta_1+1)} + \frac{1}{\Gamma(\beta_1+1)}\right) \| v(t) - v^*(t) \|$$
$$= \vartheta_1 \| v(t) - v^*(t) \|,$$

And

$$|v(t) - v^*(t)| = | \int_0^1 H^{\alpha_2}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\,\psi_2\big(\wp, u(\wp)\big)d\wp\right)ds - \int_0^1 H^{\alpha_2}(t,s)\,\phi_q\left(\int_0^1 H^{\beta_2}(s,\wp)\,\psi_2(\wp, u^*(\wp))d\wp\right)ds$$
$$\le (p-1)\sigma^{p-2}\left(\int_0^1 |H^{\alpha_2}(t,s)| \int_0^1 |H^{\beta_2}(s,\wp)|\, |\psi_2\big(\wp, u(\wp)\big) - \psi_2\big(\wp, u^*(\wp)\big)| d\wp ds\right) \tag{4.5}$$
$$\le (p-1)\sigma^{p-2}\lambda_{\psi_2}\left(\frac{1}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(m)\Gamma(\alpha_2 - m + 2)}\right)\left(\frac{1}{\Gamma(\beta_2+1)} + \frac{1}{\Gamma(\beta_2+1)}\right) \| u(t) - u^*(t) \|$$
$$= \vartheta_2 \| u(t) - u^*(t) \|$$

Where $D_1^* = \vartheta_1, D_2^* = \vartheta_2$. Hence, by the help of (4.4), (4.5) the system (3.12), (3.13), is Hyers-Ulam stable. Therefore, Eq. (1.1) is Hyers-Ulam stable.

Conclusion

We have considered a high order coupled system of FDEs with nonlinear p-Laplacian operator for the examination of existence, uniqueness of solution and Hyer-Ulam stability by using topological degree theory. For these aims, we transformed the supposed problem into an integral system via Green's function(s) and assumed certain necessary conditions over a Banach space. Our results are more general and useful than the standard case.

Author's Contributions

All the authors of the paper have equal contribution in the paper. They have read and approved the paper before submission for publication.

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