International Journal of Engineering Works
https://www.ijew.io/
ISSN-p: 2521-2419
ISSN-e: 2409-2770
Vol. 11, Issue 09, PP. 160-164 September 2024
https://doi.org/10.34259/ijew.24.1109160164

Solving System of Highly Oscillatory Ordinary Differential Equations with Residual-based Adaptive Refinement of Physics-Informed Neural Networks

Ghani Irfan[1], Haider Zaman[2], Muhammad Ishaq[3], Wajid Khan[4]
[1]Department of Basic Sciences and Islamiat University of Engineering and Technology, Peshawar, Pakistan,
[2]Department of Mathematics, University of Peshawar, Pakistan,
[3,4]Department of Mathematics, Government Superior Science College, Peshawar, Pakistan
ghani.irfan111@gmail.com[1], haider034840@gmail.com[2], m.ishaq1972@gmail.com[3], wjdkhan206@gmail.com[4]

Received: 07 August, Revised: 06 September , Accepted: 11 September

***Abstract*—** This paper presents an innovative method for solving systems of ordinary differential equations (ODEs) characterized by oscillatory solutions, utilizing Residual-Based Adaptive Refinement of Physics-Informed Neural Networks (RAR-PINNs). Conventional numerical techniques often face challenges in accurately resolving oscillatory solutions due to issues with convergence and stability. To address these challenges, we introduce a refined approach that integrates adaptive refinement strategies with physics-informed neural networks, enhancing their capability to model and predict complex oscillatory dynamics. Our method involves an adaptive mechanism that selectively refines the neural network's focus based on the residual errors of the predicted solutions, thereby improving accuracy where it is most needed. By incorporating physical constraints directly into the learning process, our approach ensures that the neural network not only captures the underlying oscillatory patterns but also adheres to the governing differential equations. We validate the effectiveness of the RAR-PINNs approach through numerical experiments on benchmark problems with known oscillatory solutions, demonstrating substantial improvements in both solution accuracy and computational efficiency compared to traditional methods. This advancement provides a powerful tool for tackling highly oscillatory ODE systems in various scientific and engineering applications where oscillatory behavior is prevalent.

***Keywords*—** Physics-Informed Neural Networks, Oscillatory Solution, System of Ordinary Differential Equations, Residual Based Adaptive Refinement,

I. INTRODUCTION

Solving systems of ordinary differential equations (ODEs) that exhibit oscillatory behavior presents significant challenges due to the intricate nature of oscillations and the limitations of traditional numerical methods. Oscillatory solutions are prevalent in various scientific and engineering problems, such as in mechanical vibrations, electrical circuits, and fluid dynamics [1], [2]. Accurate modeling and simulation of these systems are crucial for understanding complex physical phenomena and making reliable predictions.

Traditional numerical methods, such as finite difference methods or finite element methods, often struggle with oscillatory solutions due to issues with stability and convergence [3]. For instance, standard discretization techniques may suffer from numerical instability, particularly in the presence of high-frequency oscillations, which can lead to poor accuracy and excessive computational costs [4]. Moreover, these methods may require very fine discretizations to capture the oscillatory behavior accurately, which increases the computational burden.

Recent advancements in machine learning and neural networks have provided new opportunities for addressing these challenges. Physics-Informed Neural Networks (PINNs) have emerged as a powerful tool for solving differential equations by incorporating the governing physical laws directly into the learning process [5]. PINNs leverage neural networks to approximate the solution of ODEs while ensuring that the predictions satisfy the underlying differential equations. However, while PINNs have shown promise, they often face difficulties in accurately capturing oscillatory solutions due to the limitations of standard neural network architectures and training techniques.

To address these limitations, we explore an approach that combines PINNs with a Residual-Based Adaptive Refinement strategy. The method, Residual-Based Adaptive Refinement of Physics-Informed Neural Networks (RAR-PINNs), adapts the neural network's focus based on the residual errors of the predicted solutions. This adaptive refinement allows the network to concentrate its capacity on regions where the residual errors are significant, thereby improving the accuracy of the solution, especially in oscillatory regimes [6], [7]. By incorporating physical constraints directly into the learning process and refining the network's focus dynamically, RAR-

PINNs offer a more efficient and accurate approach to modeling complex ODE systems with oscillatory behavior.

In this paper, we validate the effectiveness of RAR-PINNs through numerical experiments on benchmark problems with known oscillatory solutions. Our results demonstrate that RAR-PINNs significantly improve both the accuracy of the solutions and computational efficiency compared to traditional methods and standard PINNs approaches. This advancement provides a valuable tool for researchers and engineers working on problems involving complex oscillatory dynamics.

II. METHODOLOGIES

A. *Physics-Informed Neural Networks*

Solving a system of ordinary differential equations (ODEs) using Physics-Informed Neural Networks (PINNs) involves integrating the principles of PINNs into the specific context of ODEs. Let the system of ODEs that we need to solve is

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(t, \boldsymbol{u}), \boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad (1)$$

where $\boldsymbol{u}(t)$ is a vector of state variables, and $\boldsymbol{f}$ is a vector-valued function describing the system dynamics. The domain of the problem is then divided into two types of data, training data and validation data. The next step is to describe the architecture of neural network. This step is mainly concerned with describing the number of hidden layers the number of neurons in each layer and the best choice of activation function. Number of neurons in the input layer is the number of indepdent variables involved in the system, and the number of neurons in the output layer is equal to the number of depdendent variables. However the number of neurons in the hidden layer can be increased or decresed according to the complexity of the problem. Neurons in the input layer send the input varaibles to the neurons of the hidden layer. Those neurons multiply a weight with the input variables and then add a bias with the multiple. This result is then sent to an activation function as argument. Let the resultant of each neuron be labeled as $\boldsymbol{\theta}$. The final output of the neural network will be the vector $\boldsymbol{u}_{NN}(t, \boldsymbol{\theta})$. This resultant is actually a prediction of the original function using physics-informed neural networks. Equation (1) can now be re-written using the neural prediction as

$$\frac{d\boldsymbol{u}_{NN}}{dt} = \boldsymbol{f}(t, \boldsymbol{u}_{NN}), \boldsymbol{u}_{NN}(0) = \boldsymbol{u}_{NN,0}. \qquad (2)$$

In next step the loss functions are constructed for the ODEs and the ICs as follows

$$Loss_{ODE} = |\boldsymbol{u}'_{NN} - \boldsymbol{f}(t, \boldsymbol{u}_{NN})|^2,$$

$$Loss_{IC} = \left|\boldsymbol{u}_{NN}(0) = \boldsymbol{u}_{NN,0}\right|^2.$$

Thus the total loss becomes

$$Loss_{total} = Loss_{ODE} + Loss_{IC}.$$

All the process of training depebnds on the loss function. The ultimate aim of the training will be to minimize it as much as possible by updating the weights and biases used in the output of each neuron.The stopping crieteria for the training process is userdefined and it may depend on epochs or the loss function. The flowchart of the PINNs is shown in Figure 1.

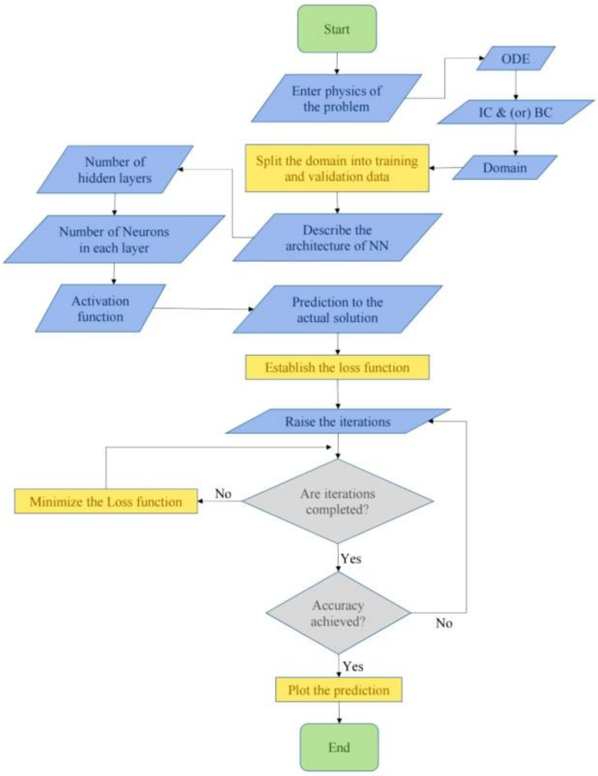

Figure 1. Flowchart of physics-informed neural networks.

B. *Residual-based Adaptive Refinement of Physics-Informed Neural Networks*

The adaptive strategy for physics-informed neural networks is considered as an advancement of the original methodology. In PINNs the training points are equally distributed over the domain. While in RAR-PINNs the more training points are added when the residual error is higher. The process involves the following steps

Step1: Train the Initial Model: Perform initial training of the PINN.

Step2: Compute Residuals: Evaluate the PDE residuals using the trained network.

Step3: Refinement: Identify high-residual regions and generate additional training points there.

Step4: Retrain: Update and retrain the network with the refined dataset.

Step5: Iterate: Repeat the process until the solution converges to the desired accuracy.

III. RESULTS AND DISCUSSIONS

In this section we are about to solve a system of highly oscillatory ordinary differential equations by both analytical method and by the method of RAR-PINNs.

Example-1

Consider the system of ordinary differential equations

$$\frac{dx}{dt} = \omega y,$$

$$\frac{dy}{dt} = -\omega x,$$

with the initial conditions $x(0) = 0$ and $y(0) = 1$. The exact solution of the system is $x(t) = \sin(\omega t)$ and $y(t) = \cos(\omega t)$.

A. Analytical Method:

Differentiate first equation with respect to t we get

$$\frac{d^2x}{dt^2} = \omega \frac{dy}{dt},$$

Substitute $\frac{dy}{dt} = -\omega x$ into this equation

$$\frac{d^2x}{dt^2} = \omega(-\omega x) = -\omega^2 x$$

This gives us the second-order differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

The characteristic equation is

$$r^2 + \omega^2 = 0.$$

Solving this we get

$$r = \pm \imath\omega.$$

The general solution to the differential equation $\frac{d^2x}{dt^2} + \omega^2 x = 0$ is $x(t) = Acos(\omega t) + Bsin(\omega t)$, where A and B are constants to be determined.

Differentiating the final result we get

$$\frac{dx}{dt} = -A\omega \sin(\omega t) + B\omega \cos(\omega t).$$

Thus,

$$\omega y = -A\omega \sin(\omega t) + B\omega\cos(\omega t).$$

Dividing by ω

$$y(t) = -Asin(\omega t) + Bcos(\omega t).$$

Use the initial conditions $x(0) = 0$ and $y(0) = 1$.

$$x(0) = Acos(0) + Bsin(0)$$

this implies that $A = 0$. So $x(t) = Bsin(\omega t)$.

Again use the initial condition for $y(t)$.

$$y(0) = -Asin(0) + Bcos(0).$$

But given that $y(0) = 1$. Thus $B = 1$. So

$$x(t) = \sin(\omega t) \text{ and } y(t) = \cos(\omega t).$$

B. RAR-PINNs

The RAR-PINNs is employed for the solution of the same problem. We considered three hidden lyaers with 50 neurons each. The number of neurons in the input layer is one due to single input variable and the number of neurons in the output layer is two because there are two dependent variables. The activation function is selected to be the hyperbolic tangent function, the Adam optimization technique is employed and the initializer is set to “Glorot Uniform”. With all these hyperparameters the RAR-PINNs method was able to give us an accuracy of 10^{-6}. The process of training took about 206.9 seconds. Figure 2 shows the plots of x and y for $\omega = 2$.

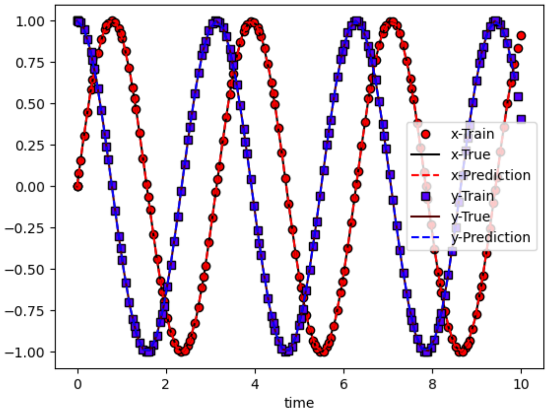

Figure 2. True vs training plots of x and y.

The mean squared error of the predicted solution to the exact solution is shown in Figure 3.

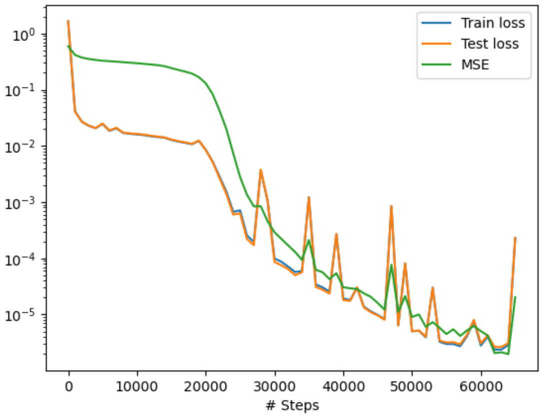

Figure 3. Steps vs MSE and losses.

From Figure 3 we can see that the training loss and the test loss along with the MSE are about 10^{-6}. Which is somehow a good accuracy.

The number of oscillations in this problem is being controlled by the parameter ω. Increasing the value of ω results in increasing number of oscillations. Let us check if the method is still prominent to predict the solution with an increased number of oscillations or no. So let us set $\omega = 3$ this time. Figure 4 shows the plot of x and y for $\omega = 3$.

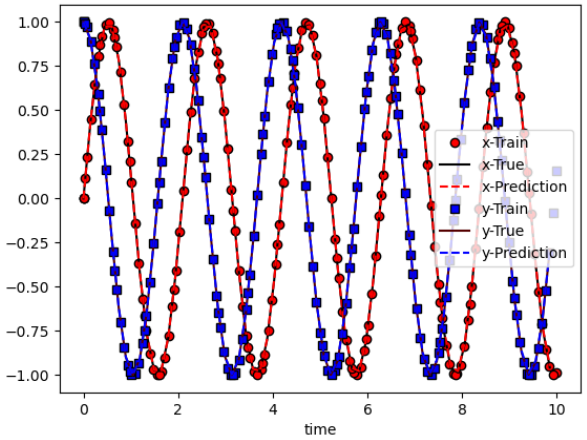

Figure 4. Example of a figure caption. (figure caption)

Comparing Figure 2 and Figure 4 one can easy understand that the increased value of ω from 2 to 3 has affected the number of oscillations. The results are obtained using the same hyperparameters. Figure 5 shows the mean squared error graph.

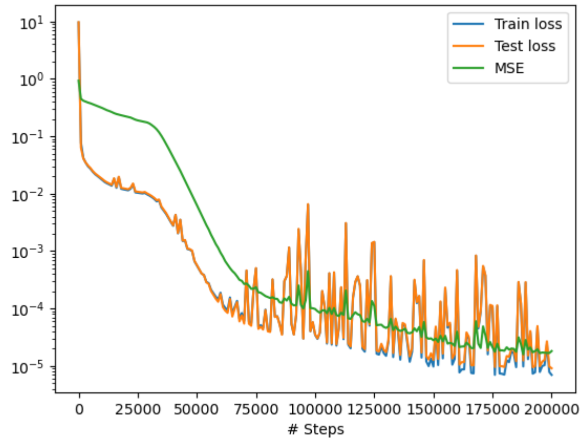

Figure 5. Steps vs MSE and losses.

From Figure 5 we see that the MSE obtained is about 10^{-5}. The time taken by the method is about 685 seconds. It simply shows that the increased number of oscillations require more time to give a better prediction.

CONCLUSION

Using the analytical method, we derived these solutions by solving the second-order differential equation resulting from differentiating the given system and applying the initial conditions.

Additionally, we employed the Residual-based Adaptive Refinement Physics-Informed Neural Networks (RAR-PINNs) to solve the same problem. We utilized a neural network with three hidden layers, each containing 50 neurons, and a hyperbolic tangent activation function. The network achieved a high accuracy of 10^{-6} with a training duration of approximately 206.9 seconds.

The results demonstrate that RAR-PINNs can effectively approximate the solution to the ODE system, with training losses and mean squared errors indicating a high degree of accuracy. Further experiments with increased ω (from 2 to 3) showed that the RAR-PINNs method continues to perform well, though with an increase in the mean squared error to 10^{-5} and a longer training time of about 685 seconds. This suggests that while the method remains accurate with higher oscillations, it requires more computational resources as the complexity of the solution increases.

Overall, the study confirms that RAR-PINNs is a robust tool for solving ODE systems with varying oscillation parameters, providing accurate solutions while adapting to increased problem complexity.

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Mr. Ghani Irfan, born on January 30, 1999, is an accomplished mathematician with a notable academic and professional background. He earned a gold medal in mathematics from Government Superior Science College, Peshawar, reflecting his exceptional proficiency in the field. Mr. Irfan has also served as a lecturer in the Department of Mathematics at Government Superior Science College, Peshawar, where he contributed to the academic development of students. He completed his Master's degree in Mathematics from the University of Engineering and Technology (UET) Peshawar, Pakistan, where he further honed his expertise. Currently, Mr. Irfan is pursuing his Ph.D.

in Mathematics at the same institution, continuing his research and academic endeavors. Mr. Irfan’s dedication to mathematics and his contributions to education reflect his commitment to advancing the field and supporting the academic community.

How to cite this article:
Ghani Irfan, Haider Zaman, Muhammad Ishaq, Wajid Khan “Solving System of Highly Oscillatory Ordinary Differential Equations with Residual-based Adaptive Refinement of Physics-Informed Neural Networks” International Journal of Engineering Works, Vol. 11, Issue 09, PP. 160-164, September 2024. https://doi.org/10.34259/ijew.24.1109160164.